

Question 5

The pair $\cos t, \sin t$ is fundamental for $y'' + y = 0$ on $\mathbb{R}$. Fix $L > 0$ and prescribe two position measurements

$$y(0) = a, \quad y(L) = b, \quad a, b \in \mathbb{R}.$$

Tasks

1. Find the coefficient equations and determine when the measurements select a unique solution.
2. For every exceptional length, give necessary and sufficient conditions for existence and describe all solutions when they exist.
3. At $L = \pi$, describe all solutions for $(a, b) = (1, -1)$ and decide whether any solution exists for $(a, b) = (1, 1)$.
4. Explain why the exceptional lengths do not mean that the fundamental pair has become dependent. Identify which additional measurement at zero would uniquely determine a solution.

Question 5 – Solution

Strategy. Distinguish a fundamental set of functions from the invertibility of a particular measurement map.

Step 1: Solve at ordinary lengths. Every solution is $A \cos t + B \sin t$. The position measurements give

$$A = a, \quad a \cos L + B \sin L = b.$$

If $\sin L \neq 0$, the unique solution is

$$y = a \cos t + \frac{b - a \cos L}{\sin L} \sin t.$$

Thus uniqueness holds exactly at lengths that are not positive integer multiples of π .

Step 2: Classify exceptional data. If $L = n\pi$, $n = 1, 2, \dots$, the second equation reduces to $b = (-1)^n a$. If this fails, there is no solution. If it holds, B is arbitrary and all solutions are $a \cos t + B \sin t$; there are infinitely many.

Step 3: Apply the classification. At $L = \pi$, the data $(1, -1)$ give

$$y = \cos t + B \sin t, \quad B \in \mathbb{R}.$$

The data $(1, 1)$ fail the necessary condition $b = -a$ and give no solution. The figure shows three members of the compatible family.

Step 4: Explain what failed. The data vectors of $\cos t, \sin t$ using value and slope at zero are $(1, 0)$ and $(0, 1)$, so the pair remains fundamental. At $L = n\pi$, both endpoint positions are insensitive to the coefficient B . Measuring $y'(0) = c$ fixes $B = c$ uniquely; the endpoint data must still be compatible with that solution.

