

Question 4

The smooth functions $u = t^2$ and $v = t^3$ are proposed as a fundamental set for an equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients on $\mathbb{R}$.

Tasks

1. Prove that u, v are linearly independent as functions on $\mathbb{R}$.
2. Prove that no such regular normalized equation on $\mathbb{R}$ can have both as solutions. Explain why independence alone is insufficient.
3. Find the normalized equation having both as solutions on $(0, \infty)$ and verify that they form a fundamental set there.
4. On $(0, \infty)$, solve $y(1) = 0$, $y'(1) = 1$ using the pair. Explain why its zero value and slope at zero do not contradict your conclusion about regular equations.

Question 4 – Solution

Strategy. Check both requirements: independence and membership in the solution space of one regular equation on the stated interval.

Step 1: Prove independence. If $At^2 + Bt^3 = 0$ for every real t , then $A + Bt = 0$ for every nonzero t . Evaluating at two distinct nonzero points forces $A = B = 0$.

Step 2: Reject the proposed regular equation. The nonzero function $u = t^2$ has $u(0) = u'(0) = 0$. Uniqueness for continuous normalized coefficients would force it to equal the zero solution on $\mathbb{R}$. Equivalently, substitution at zero would require $u''(0) = 2 = 0$. Thus these independent functions cannot be a fundamental set for any equation of the stated kind.

Step 3: Work on a valid interval. Substituting the two functions for $t > 0$ gives

$$2 + 2tp + t^2q = 0, \quad 6t + 3t^2p + t^3q = 0.$$

Subtracting t times the first equation from the second yields $p = -4/t$, then $q = 6/t^2$. Hence

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0.$$

Both residuals vanish, the coefficients are continuous on $(0, \infty)$, and $uv' - u'v = t^4 > 0$ there, so the pair is fundamental.

Step 4: Fit and interpret. The data give $A + B = 0$, $2A + 3B = 1$, so $y = t^3 - t^2$. Its polynomial extension has zero value and slope at zero, but the normalized equation is undefined there. A theorem requiring continuous coefficients on an interval containing zero cannot be applied to that extension.

