

Question 3

For $y'' - 3y' + 2y = 0$ on $\mathbb{R}$, let

$$f = e^t, \quad g = e^{2t}, \quad h = f + g, \quad k = 2f - g.$$

A student writes $y = Af + Bg + Ch + Dk$ and claims this gives four independent solution freedoms.

Tasks

1. Verify that all four functions solve the equation and that f, g form a fundamental set.
2. Determine which of the six unordered pairs chosen from f, g, h, k are fundamental. Justify every pair.
3. Find every quadruple (A, B, C, D) that represents the zero function, and explain the error in the student's claim.
4. Find all quadruples representing the solution with $y(0) = 1$, $y'(0) = 3$. Give its unique representation using only h, k .

Question 3 – Solution

Strategy. Express every candidate in the already verified pair f, g , then count effective coefficients.

Step 1: Verify the underlying pair. Substitution of e^{rt} gives residual $(r^2 - 3r + 2)e^{rt}$, which vanishes for $r = 1, 2$. Linearity verifies h, k . The data determinant for f, g at zero is $2 - 1 = 1$, so this pair is fundamental.

Step 2: Check all six pairs. The coefficient columns relative to f, g are $(1, 0)$, $(0, 1)$, $(1, 1)$ and $(2, -1)$. Their determinants, in the listed order, are

pair	(f, g)	(f, h)	(f, k)	(g, h)	(g, k)	(h, k)
determinant	1	1	-1	-1	-2	-3

Every determinant is nonzero, so every pair is an invertible change of the fundamental pair f, g and is itself fundamental.

Step 3: Identify redundancy. The proposed expression is

$$y = (A + C + 2D)f + (B + C - D)g.$$

It is zero exactly when $A = -C - 2D$ and $B = -C + D$, where C, D are arbitrary. These nontrivial constant relations show that the four functions are not independent. Only two effective coefficients are present.

Step 4: Fit the specified data. Writing $y = af + bg$ gives $a + b = 1$, $a + 2b = 3$, hence $a = -1, b = 2$. All four-coefficient representations are

$$(A, B, C, D) = (-1 - C - 2D, 2 - C + D, C, D).$$

Since $h - k = -f + 2g$, the unique representation in the pair h, k is $y = h - k$. Its data are $-1 + 2 = 1$ and $-1 + 4 = 3$.