

Question 2

For the equation $y'' - y = 0$ on $\mathbb{R}$, consider the parameter-dependent pair

$$u = e^t + e^{-t}, \quad v_\alpha = e^t + \alpha e^{-t}, \quad \alpha \in \mathbb{R}.$$

Coefficients in a fundamental set must be constant in t .

Tasks

1. Verify both solutions and find all α for which the pair is fundamental.
2. For those parameters, find the coefficients of the solution with $y(0) = 0$, $y'(0) = 1$ in this pair.
3. As $\alpha \rightarrow 1$, compare the behavior of these coefficients with the behavior of the represented solution.
Can large coefficients alone establish large solution values?
4. Construct from u, v_α a companion with data $(0, 1)$ that extends to a well-defined function at $\alpha = 1$.
Explain why simply setting $\alpha = 1$ in the original pair fails.

Question 2 – Solution

Strategy. Track the initial-data vectors as the two proposed basis elements approach dependence.

Step 1: Test independence. Each exponential solves $y'' - y = 0$, so both combinations do. The data vectors at zero are $(2, 0)$ and $(1 + \alpha, 1 - \alpha)$, with determinant $2(1 - \alpha)$. The pair is fundamental exactly when $\alpha \neq 1$. At 1, both functions equal u .

Step 2: Fit the data. In $y = Au + Bv_\alpha$, the data require

$$2A + (1 + \alpha)B = 0, \quad (1 - \alpha)B = 1.$$

Thus

$$A = -\frac{1 + \alpha}{2(1 - \alpha)}, \quad B = \frac{1}{1 - \alpha}.$$

Step 3: Simplify before taking a limit. Substitution cancels the large terms exactly:

$$Au + Bv_\alpha = \frac{1}{2}(e^t - e^{-t}) = \sinh t.$$

Both coefficients grow without bound in magnitude near $\alpha = 1$, while the function is independent of α . Large coordinates in a nearly dependent pair do not imply large values of the function on a fixed interval.

Step 4: Replace the collapsing direction. Define, for $\alpha \neq 1$,

$$z_\alpha = \frac{v_\alpha - (1 + \alpha)u/2}{1 - \alpha} = \sinh t.$$

This extends by $z_1 = \sinh t$ and has data $(0, 1)$. The pair u, z_1 remains fundamental. The original pair at $\alpha = 1$ contains only one effective direction and cannot supply the requested nonzero slope.

