

Question 1

A fundamental set for a homogeneous linear equation on an interval is a pair of solutions whose constant linear combinations represent every solution uniquely. Let p, q be continuous on an open interval I . For two solutions of $y'' + py' + qy = 0$, define

$$D(t_0) = u(t_0)v'(t_0) - u'(t_0)v(t_0), \quad t_0 \in I.$$

Tasks

1. Using existence and uniqueness for initial data, prove that the pair is fundamental on I if and only if $D(t_0) \neq 0$.
2. For $y'' + 2y' + 5y = 0$ on $\mathbb{R}$, verify $u = e^{-t} \cos 2t$, $v = e^{-t} \sin 2t$ and apply the criterion at zero.
3. Construct solutions ϕ, ψ with initial data $(\phi(0), \phi'(0)) = (1, 0)$ and $(\psi(0), \psi'(0)) = (0, 1)$.
4. Use this normalized pair to solve $y(0) = 2$, $y'(0) = -3$, and verify the data.

Question 1 – Solution

Strategy. A fundamental pair must supply every value-and-slope pair, so test the two-by-two initial-data system.

Step 1: Prove the criterion. If $D(t_0) \neq 0$, the equations

$$Au(t_0) + Bv(t_0) = a, \quad Au'(t_0) + Bv'(t_0) = b$$

have unique constants A, B for every a, b . The resulting combination solves the equation; uniqueness makes it equal to every solution with those data. Conversely, if $D(t_0) = 0$, some nonzero constant pair (A, B) gives zero value and slope. By uniqueness, $Au + Bv$ is identically zero, contradicting unique representation. Thus the criterion is necessary and sufficient.

Step 2: Verify the candidates. Their derivatives are

$$u' = e^{-t}(-\cos 2t - 2\sin 2t), \quad u'' = e^{-t}(-3\cos 2t + 4\sin 2t),$$

$$v' = e^{-t}(-\sin 2t + 2\cos 2t), \quad v'' = e^{-t}(-3\sin 2t - 4\cos 2t).$$

Both residuals vanish. At zero their data are $(1, -1)$ and $(0, 2)$, so $D(0) = 2 \neq 0$ and the pair is fundamental on $\mathbb{R}$.

Step 3: Normalize the data. The required choices are

$$\phi = u + \frac{1}{2}v, \quad \psi = \frac{1}{2}v.$$

Their data are $(1, 0)$ and $(0, 1)$ respectively, so they too form a fundamental set.

Step 4: Fit the solution. The normalized coefficients equal the data:

$$y = 2\phi - 3\psi = e^{-t}(2\cos 2t - \frac{1}{2}\sin 2t).$$

At zero its value is 2; its slope is $-2 - 1 = -3$. It satisfies the equation by linearity and is unique on $\mathbb{R}$.