

Question 10

On $t > 0$, consider the forced equation

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = t, \quad y(1) = 0, \quad y'(1) = 0.$$

The function $y_1 = t$ is supplied as a solution of the *associated homogeneous equation*.

Tasks

1. Verify the stated role of y_1 and explain why it is not a solution of the forced equation.
2. Substitute $y = tv$ and derive a reduced equation. Find the full forced solution family, retaining both integration constants.
3. Solve the given initial-value problem.
4. Check the original residual and data directly. Explain the error in applying the homogeneous reduction formula without carrying along the forcing term.

Question 10 – Solution

Strategy. The known homogeneous solution still cancels the term containing v , but the nonzero right-hand side remains.

Step 1: Verify the seed's role. For $y_1 = t$, the left-hand side is $0 - 2/t + 2/t = 0$. This satisfies the homogeneous equation, but cannot equal the positive forcing t on the specified domain.

Step 2: Reduce with the forcing present. With $y = tv$,

$$y' = v + tv', \quad y'' = 2v' + tv''.$$

Substitution cancels the terms in v and v' :

$$2v' + tv'' - \frac{2}{t}(v + tv') + \frac{2}{t}v = tv'' = t.$$

Since $t > 0$, $v'' = 1$. Equivalently the first-order reduced equation is $w' = 1$, with $w = v'$. Integrating twice gives

$$v = \frac{1}{2}t^2 + At + B, \quad \boxed{y = \frac{1}{2}t^3 + At^2 + Bt.}$$

Step 3: Impose the data. At 1 the equations are $1/2 + A + B = 0$ and $3/2 + 2A + B = 0$. Thus $A = -1$, $B = 1/2$, yielding

$$\boxed{y = \frac{1}{2}t(t-1)^2.}$$

Step 4: Verify and diagnose. In expanded form the solution is $t^3/2 - t^2 + t/2$, with $y' = 3t^2/2 - 2t + 1/2$ and $y'' = 3t - 2$. Its left-hand side is

$$(3t - 2) + (-3t + 4 - 1/t) + (t - 2 + 1/t) = t.$$

At 1 its value and slope are zero. Discarding the forcing would instead give $w' = 0$, producing only $At^2 + Bt$, whose left-hand side is zero. Those two terms provide the homogeneous freedom, but the cubic term is essential to match the forcing.