

Question 9

Consider the original undivided equation $t^2 y'' - 6y = 0$. First work on $t > 0$ with known solution $y_1 = t^3$. A classical solution through zero means a C^2 function satisfying the undivided equation on an open interval containing zero.

Tasks

1. Use reduction of order to find every solution on $t > 0$.
2. Which right-hand solutions admit a C^2 extension through zero? Find the forced value, slope and second derivative at zero.
3. For each admissible right-hand solution, classify all classical continuations to $t < 0$. Explain whether the zero initial data determine a unique continuation.
4. Which of those continuations are C^3 at zero? Explain why the usual uniqueness theorem does not settle this problem.

Question 9 – Solution

Strategy. Solve on each side of the singular coefficient, then compare one-sided derivatives at the join.

Step 1: Reduce on the positive side. The seed has residual $t^2(6t) - 6t^3 = 0$. In normalized form $p = 0$, so

$$v' = Ct^{-6}, \quad v = D - \frac{1}{5}Ct^{-5}, \quad \boxed{y = At^3 + Bt^{-2} \quad (t > 0)}.$$

These two powers each satisfy the original equation by direct substitution.

Step 2: Exclude the divergent mode. If $B \neq 0$, then y diverges at zero, so even a continuous extension is impossible. If $B = 0$, the limits of y, y', y'' are $0, 0, 0$. These are the required values for a C^2 extension.

Step 3: Classify continuations. On $t < 0$ the same calculation gives $at^3 + bt^{-2}$; continuity forces $b = 0$. Thus all classical continuations of the right branch At^3 are

$$y(t) = \begin{cases} at^3, & t < 0, \\ 0, & t = 0, \\ At^3, & t > 0, \end{cases} \quad a \in \mathbb{R}.$$

The first and second derivatives match at zero for every a . At zero the undivided residual is $0 - 6(0) = 0$. Therefore these are sufficient as well as necessary, and infinitely many continuations share $y(0) = y'(0) = 0$ even with the right branch fixed.

Step 4: Require one more derivative. The one-sided third derivatives are $6a$ and $6A$; a C^3 join occurs exactly when $a = A$. In that case the single polynomial At^3 holds on both sides. The leading coefficient is zero at the joining point, and the normalized coefficient $-6/t^2$ is undefined there, so the regular uniqueness theorem has no applicable interval containing zero.