

Question 8

On $\mathbb{R}$, solve the boundary-at-infinity problem

$$y'' + \frac{2t}{1+t^2}y' = 0, \quad \lim_{t \rightarrow -\infty} y(t) = -1, \quad \lim_{t \rightarrow \infty} y(t) = 3.$$

The constant function $y_1 = 1$ is a known homogeneous solution.

Tasks

1. Use reduction of order to obtain the full solution family.
2. Find the unique solution satisfying the two limiting conditions and verify its equation.
3. Find its value and slope at zero, monotonicity and exact range. Is every solution of this equation bounded?
4. Compute $\lim_{t \rightarrow \infty} t[3 - y(t)]$. Decide whether constants $M, c > 0$ can satisfy $3 - y(t) \leq Me^{-ct}$ for all sufficiently large t .

Question 8 – Solution

Strategy. A constant seed reduces the equation directly to a first-order equation for the slope.

Step 1: Integrate the slope. Set $w = y'$. An integrating factor is $1 + t^2$, so

$$((1 + t^2)w)' = 0, \quad w = B/(1 + t^2), \quad y = A + B \arctan t.$$

Step 2: Fit the two limits. The equations $A - B\pi/2 = -1$ and $A + B\pi/2 = 3$ give

$$y = 1 + \frac{4}{\pi} \arctan t.$$

This is the unique choice of the two constants. Its derivatives $4/[\pi(1 + t^2)]$ and $-8t/[\pi(1 + t^2)^2]$ cancel in the original equation.

Step 3: Describe the trajectory. The value and slope at zero are 1 and $4/\pi$. The slope is positive everywhere; continuity and the limiting values give range $(-1, 3)$. Every member $A + B \arctan t$ is bounded, since $|\arctan t| < \pi/2$.

Step 4: Determine the tail rate. For $t > 0$,

$$3 - y(t) = \frac{4}{\pi} \arctan(1/t), \quad \lim_{t \rightarrow \infty} t[3 - y(t)] = \frac{4}{\pi}.$$

Here $\arctan u/u \rightarrow 1$ as $u \rightarrow 0$. An exponential bound would force this positive limit to be at most $\lim Mte^{-ct} = 0$, a contradiction. The approach to the limit has an algebraic $1/t$ tail.

