

Question 7

On $\mathbb{R}$, consider $(1+t^2)y'' - 2y = 0$ with known solution $y_1 = 1+t^2$. Let y_2 denote the solution with $y_2(0) = 0$, $y_2'(0) = 1$, and define $I = y_2/(1+t^2)$.

Tasks

1. Use reduction of order to find y_2 explicitly. Evaluate the required integral.
2. Verify the equation and the two normalization data directly.
3. Find the exact range and monotonicity of $I(t)$ on $\mathbb{R}$, including the limiting values at infinity.
4. For every real C , classify the finite zeros and the sign of $y_C = C(1+t^2) - y_2$. Include the two boundary parameter values.

Question 7 – Solution

Strategy. Divide the solution family by its positive seed so that zeros become level crossings of a monotone function.

Step 1: Construct the companion. The seed residual is $2(1+t^2) - 2(1+t^2) = 0$. Since $p = 0$,

$$I(t) = \int_0^t \frac{ds}{(1+s^2)^2} = \frac{t}{2(1+t^2)} + \frac{1}{2} \arctan t.$$

The integral follows by setting $s = \tan u$ and integrating $\cos^2 u$. Hence

$$y_2 = \frac{1}{2}t + \frac{1}{2}(1+t^2) \arctan t.$$

Step 2: Verify. We obtain $y_2' = 1 + t \arctan t$ and $y_2'' = \arctan t + t/(1+t^2) = 2y_2/(1+t^2)$. The equation holds and the data are 0, 1.

Step 3: Determine the ratio range. Since $I' = 1/(1+t^2)^2 > 0$, the ratio is strictly increasing. Its limits are $-\pi/4$ and $\pi/4$; neither is attained. Its range is exactly $(-\pi/4, \pi/4)$ by continuity.

Step 4: Classify all parameters. The sign of y_C is the sign of $C - I(t)$. If $|C| < \pi/4$, there is exactly one finite zero, with positive values before it and negative values after it. If $C \geq \pi/4$, the solution is strictly positive everywhere; if $C \leq -\pi/4$, it is strictly negative everywhere. Equality in these last two cases does not create a finite zero because the limiting ratio levels are excluded.

