

Question 6

On $t > 0$, consider

$$ty'' - (t+1)y' + y = 0, \quad y_1 = e^t.$$

A student inserts the undivided coefficient $-(t+1)$ into the usual normalized reduction formula and obtains $v' = e^{t^2/2-t}$.

Tasks

1. Verify the seed, normalize the equation, and derive the correct equation for $w = v'$.
2. Find the full solution family and solve $y(1) = 0$, $y'(1) = 1$.
3. Compute the residual of the student's candidate $y = e^t v$, with $v' = e^{t^2/2-t}$. Explain why testing only $t = 1$ would miss the error.
4. For the original undivided equation at $t = 0$, find the necessary relation between value and slope. Exhibit two distinct smooth solutions with zero value and slope there.

Question 6 – Solution

Strategy. The coefficient in the standard formula is the coefficient after division by the leading term.

Step 1: Normalize. For e^t , the residual is $[t - (t + 1) + 1]e^t = 0$. Division gives $p = -1 - 1/t$, so

$$w' + (1 - 1/t)w = 0, \quad w = Cte^{-t}.$$

Step 2: Integrate and fit. Since $\int te^{-t} dt = -(t + 1)e^{-t}$,

$$y = Ae^t + B(t + 1).$$

At 1 the data give $Ae + 2B = 0$ and $Ae + B = 1$. Thus $B = -1$, $A = 2/e$, and

$$y = 2e^{t-1} - (t + 1).$$

Both basis functions have zero residual, and substitution at 1 gives 0, 1.

Step 3: Diagnose the error. For any v , the undivided residual of $e^t v$ is

$$e^t[tw' + (t - 1)w].$$

The student's w satisfies $w' = (t - 1)w$, giving residual

$$e^t(t^2 - 1)e^{t^2/2-t}.$$

This is zero at $t = 1$ but equals $3e^2$ at $t = 2$. A single vanishing residual does not establish a differential identity.

Step 4: Examine the singular initial point. At zero the original equation requires $y'(0) = y(0)$. Both $y = 0$ and $y = e^t - (t + 1)$ are smooth solutions through zero with value and slope zero; the second is nonzero, for example because its second derivative at zero is 1. The regular initial-value theorem fails here because the leading coefficient vanishes. Compatibility does not imply uniqueness.