

Question 5

On $t > 0$, consider $t^2 y'' - 2ty' + 2y = 0$ with seed $y_1 = t$. Use the following *unnormalized* companion convention:

$$Y_a(t) = y_1(t) \int_a^t \frac{s^2}{y_1(s)^2} ds, \quad a > 0.$$

The numerator s^2 is held fixed when changing a or rescaling the seed.

Tasks

1. Verify the seed and derive the kernel in this convention from the normalized equation.
2. Compute Y_1 , Y_2 , and the companion obtained with seed $3t$ and lower limit 1.
3. Explain why these choices give the same solution space. Why does an arbitrary additive integration constant not create a third solution freedom?
4. Use $y = At + BY_1(t)$ to solve $y(1) = 2$, $y'(1) = 1$. Verify the equation and data.

Question 5 – Solution

Strategy. Compare the actual functions produced by changes in normalization, rather than counting written constants.

Step 1: Identify the kernel. The seed residual is $-2t + 2t = 0$. After division by t^2 , $p = -2/t$ and

$$P(t) = \int_1^t -2/s \, ds = -2 \ln t, \quad e^{-P(t)} = t^2.$$

Reduction gives $v' = Ct^2/y_1^2$, which yields the specified convention.

Step 2: Compute three companions. For $y_1 = t$, the integrand equals 1, so

$$Y_1 = t(t-1) = t^2 - t, \quad Y_2 = t(t-2) = t^2 - 2t.$$

For seed $3t$, the integrand is $1/9$, hence

$$\tilde{Y}_1 = 3t(t-1)/9 = (t^2 - t)/3.$$

Step 3: Compare freedoms. We have $Y_2 = Y_1 - t$ and $\tilde{Y}_1 = Y_1/3$. Adding a seed multiple or multiplying a companion by a nonzero constant leaves the family unchanged: every choice spans $\{At + Bt^2 : A, B \in \mathbb{R}\}$. An additive constant inside the antiderivative produces only another multiple of the seed, already present in the family.

Step 4: Fit and verify. Since $Y_1(1) = 0$, $Y_1'(1) = 1$, the data give $A = 2$ and $A + B = 1$. Therefore

$$\boxed{y = 2t - (t^2 - t) = 3t - t^2.}$$

Its derivatives are $3 - 2t$ and -2 , and

$$t^2(-2) - 2t(3 - 2t) + 2(3t - t^2) = 0.$$

At $t = 1$ the value is 2 and the slope is 1. Two effective constants are sufficient and necessary for arbitrary regular initial data.