

Question 4

For $y'' + y = 0$, suppose the known solution is $y_1 = \sin t$. First work on $(0, \pi)$, where the seed does not vanish.

Tasks

1. Use $y = (\sin t)v$ to construct a second solution on $(0, \pi)$.
2. Explain whether the poles of the quotient v prevent the second solution from extending to $\mathbb{R}$.
3. Use the extended solutions to solve $y(0) = 1$, $y'(0) = 0$, and justify uniqueness.
4. A proposed continuation is $\cos t + a \sin t$ for $t < 0$ and $\cos t + b \sin t$ for $t > 0$, with $y(0) = 1$. Find the conditions for continuity, for C^1 regularity, and for being a classical solution through zero.

Question 4 – Solution

Strategy. Separate a failure of division by the seed from a singularity of the original equation.

Step 1: Reduce on the valid interval. Substitution gives

$$v'' + 2 \cot t v' = 0, \quad v' = C \csc^2 t, \quad v = -C \cot t + D.$$

Thus $y = -C \cos t + D \sin t$; a second solution is $\cos t$.

Step 2: Extend the product. Although $\cos t / \sin t = \cot t$ has poles at multiples of π , its product with $\sin t$ is $\cos t$ on each valid interval. The latter is smooth and directly satisfies $y'' + y = 0$ on all of $\mathbb{R}$. The poles belong to the chosen quotient.

Step 3: Solve at the seed's zero. In $y = A \cos t + B \sin t$, the data give $A = 1$, $B = 0$. Thus $\boxed{y = \cos t}$. Continuous coefficients and nonzero leading coefficient ensure global uniqueness for the data at zero.

Step 4: Match derivatives. Both one-sided values are 1, so continuity holds for all real a, b . The one-sided slopes are a, b , so C^1 regularity requires $a = b$. In that case one smooth formula holds on both sides, and it is a classical solution. If $a \neq b$, a derivative at zero does not exist; matching only position is insufficient.

