

Question 3

Consider the equation on $\mathbb{R}$

$$y'' - 2ty' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

with known solution $y_1 = e^{t^2}$. An integral is an acceptable exact answer.

Tasks

1. Verify the seed and derive the reduced first-order equation.
2. Find the initial-value solution as a definite integral and verify its residual without evaluating the integral.
3. Prove that the solution is odd and strictly increasing on $\mathbb{R}$.
4. Prove $t \leq y(t) \leq te^{t^2}$ for $t \geq 0$ and compute $y(1)$ to six decimal places. Explain why an unevaluated integral still defines a valid solution.

Question 3 – Solution

Strategy. Keep the definite integral and use the fundamental theorem of calculus for verification and bounds.

Step 1: Reduce. The seed has derivatives $2te^{t^2}$ and $(2 + 4t^2)e^{t^2}$, so its residual vanishes. With $y = e^{t^2}v$,

$$w' + 2tw = 0, \quad w = v' = Ce^{-t^2}.$$

Step 2: Fit and verify. The zero initial value sets the constant in v to zero; the initial slope sets $C = 1$. Therefore

$$y(t) = e^{t^2} \int_0^t e^{-s^2} ds.$$

Differentiation gives $y' = 2ty + 1$, then $y'' = 2y + 2ty'$. The residual is zero, and at zero the value and slope are 0, 1.

Step 3: Use symmetry and sign. The integrand is positive and even, so its integral from zero is odd and has the sign of t . Multiplication by e^{t^2} preserves both properties. Consequently $ty \geq 0$ and $y' = 2ty + 1 \geq 1$ everywhere: y is strictly increasing.

Step 4: Bound and evaluate. For $0 \leq s \leq t$,

$$e^{-t^2} \leq e^{-s^2} \leq 1 \implies te^{-t^2} \leq \int_0^t e^{-s^2} ds \leq t.$$

Multiplying by e^{t^2} proves the bounds. Numerical integration gives $y(1) \approx 2.030078$. The integrand is smooth, so the definite integral defines a smooth function on all of $\mathbb{R}$; the exact residual check proves it solves the equation.

