

Question 2

On $t > 0$, consider

$$t^2 y'' - 3ty' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 3.$$

A proposed known solution is $y_1 = t^2$.

Tasks

1. Verify y_1 and use reduction of order to find the general solution.
2. Solve the initial-value problem and verify both data.
3. Find every zero and stationary point of this solution on $t > 0$, and classify the stationary point.
4. Determine whether this solution has a C^1 extension and whether it has a C^2 extension through $t = 0$. Justify using one-sided limits.

Question 2 – Solution

Strategy. Normalize the equation before reduction, then use exact derivatives to study the singular endpoint.

Step 1: Find the missing solution. For t^2 , the residual is $(2 - 6 + 4)t^2 = 0$. With $y = t^2v$, the divided equation reduces to

$$v'' + \frac{1}{t}v' = 0, \quad v' = D/t, \quad \boxed{y = t^2(C + D \ln t)}.$$

Step 2: Impose the data. At 1, $y = C$ and $y' = 2C + D$. Thus $C = D = 1$ and

$$\boxed{y = t^2(1 + \ln t)}, \quad y' = t(3 + 2 \ln t), \quad y'' = 5 + 2 \ln t.$$

These give $y(1) = 1$ and $y'(1) = 3$.

Step 3: Locate the features. The sole zero is $t = e^{-1}$. The derivative changes from negative to positive at $t = e^{-3/2}$, giving the unique global minimum

$$(t, y) = (e^{-3/2}, -\frac{1}{2}e^{-3}).$$

Step 4: Test endpoint regularity. As $t \downarrow 0$, both y and y' tend to zero, but $y'' \rightarrow -\infty$. Setting $y = 0$ for $t \leq 0$ gives a C^1 extension (the derivative at zero is also $\lim y(t)/t = 0$). No C^2 extension is possible. The leading coefficient vanishes at zero, so the regular normalized theorem does not apply there.

