

Question 1

Let p, q be continuous on an open interval I , let $t_0 \in I$, and let y_1 be a known solution of

$$y'' + p(t)y' + q(t)y = 0$$

that never vanishes on I . Write $P(t) = \int_{t_0}^t p(s) ds$.

Tasks

1. Derive the equation for $w = v'$ after substituting $y = y_1 v$.
2. Solve the first-order equation for w and give an integral formula for y .
3. Construct a solution z with $z(t_0) = 0$ and $z'(t_0) = 1$. Prove that z is not a constant multiple of y_1 .
4. Express the solution with $y(t_0) = a$, $y'(t_0) = b$ in terms of y_1, z . Explain why the nonvanishing hypothesis matters.

Question 1 – Solution

Strategy. Factor out the known solution, then normalize the new solution by its initial slope.

Step 1: Reduce the order. Product differentiation gives

$$y' = y_1'v + y_1v', \quad y'' = y_1''v + 2y_1'v' + y_1v''.$$

The coefficient of v is zero because y_1 solves the equation. Hence

$$y_1v'' + (2y_1' + py_1)v' = 0, \quad w' + (2y_1'/y_1 + p)w = 0.$$

Step 2: Integrate twice. An integrating factor is $y_1^2e^P$, so

$$w = C \frac{e^{-P}}{y_1^2}, \quad y = y_1(t)[A + C \int_{t_0}^t \frac{e^{-P(s)}}{y_1(s)^2} ds].$$

Both integrations contribute a constant. The formula includes $w = 0$.

Step 3: Normalize the companion. Set

$$z(t) = y_1(t_0)y_1(t) \int_{t_0}^t \frac{e^{-P(s)}}{y_1(s)^2} ds.$$

The integral vanishes at t_0 . Since $P(t_0) = 0$, differentiation gives $z'(t_0) = 1$. If $z = ky_1$, its zero value forces $k = 0$, contradicting this slope.

Step 4: Match arbitrary data. The two constants are fixed by

$$y = \frac{a}{y_1(t_0)}y_1 + [b - \frac{ay_1'(t_0)}{y_1(t_0)}]z.$$

This is the unique solution on I by the regular linear initial-value theorem. Division by y_1 and the integral kernel require $y_1 \neq 0$. A zero of the seed may obstruct this substitution without obstructing the original equation.