

Question 10

A dimensionless family of repeated-root responses satisfies

$$y'' + 2\lambda y' + \lambda^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad \lambda > 0.$$

The design requirement is $y(t) \leq 0.05$ for every $t \geq 2$, with the additional constraint $|y''(0)| \leq 4$.

Tasks

1. Solve the IVP and prove that the response is positive and strictly decreasing for $t > 0$.
2. Show that the first time at level 0.05 is z_*/λ , where $z_* > 0$ is the unique root of $(1+z)e^{-z} = 0.05$. Locate z_* between two consecutive four-decimal-place numbers by a justified numerical calculation.
3. Translate both the settling requirement and the initial-acceleration constraint into restrictions on λ . Determine whether they can be satisfied simultaneously.
4. Find the earliest achievable settling time under the acceleration constraint alone. Sketch the responses for $\lambda = 1, 2$ with the tolerance and required time marked, and explain why simply increasing λ is not an admissible unlimited improvement.

Question 10 – Solution

Strategy. Scale time by the repeated decay rate, then compare the performance requirement with the independent constraint on the initial acceleration.

Step 1: Solve and establish monotonicity. The repeated root is $-\lambda$. The data give

$$\boxed{y = (1 + \lambda t)e^{-\lambda t}}, \quad y' = -\lambda^2 t e^{-\lambda t}.$$

The response is positive, strictly decreasing for $t > 0$, starts at 1 and tends to zero. Therefore crossing the tolerance once is enough to guarantee it thereafter.

Step 2: Reduce the crossing calculation to one dimensionless number. Let $f(z) = (1 + z)e^{-z}$ for $z \geq 0$. Then $f(0) = 1$, $f(z) \rightarrow 0$ and $f'(z) = -ze^{-z} < 0$ for $z > 0$. The intermediate value theorem and strict decrease give exactly one positive root $f(z_*) = 0.05$.

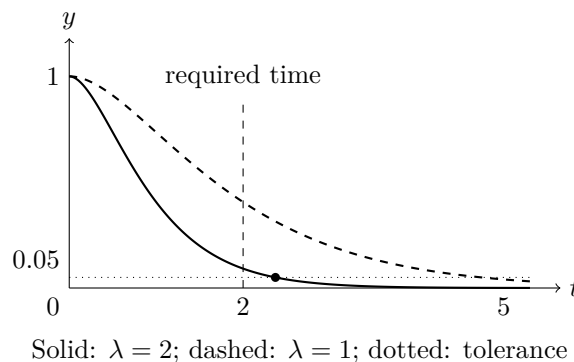
Direct evaluation gives $f(4.7438) > 0.05$ and $f(4.7439) < 0.05$. Thus

$$\boxed{4.7438 < z_* < 4.7439}, \quad t_{\text{settle}} = z_*/\lambda.$$

Numerically $z_* \approx 4.743865$; the bracket, rather than a rounded equality, supports the feasibility comparisons below.

Step 3: Test simultaneous feasibility. Settling by time 2 requires $z_*/\lambda \leq 2$, or $\lambda \geq z_*/2 > 2.3719$. But the equation at zero gives $y''(0) = -\lambda^2$, so the acceleration constraint requires $\lambda \leq 2$. These ranges do not intersect: the two requirements are incompatible.

At the largest admissible value $\lambda = 2$, the response at the deadline is $y(2) = 5e^{-4} \approx 0.091578 > 0.05$, confirming the failure directly.



Step 4: Optimize within the allowed family. The settling time z_*/λ strictly decreases as λ increases. Under $0 < \lambda \leq 2$, it is minimized at $\lambda = 2$, giving

$$\boxed{t_{\min} = z_*/2 \approx 2.371932}.$$

This is later than the required time 2. Larger λ would improve the settling time but violate the initial-acceleration bound. The conclusion optimizes this explicitly specified family and constraint, not every possible differential-equation model or control law.