

Question 9

For $L > 0$ and strictly positive endpoint values a, b , consider

$$y'' + 2y' + y = 0, \quad y(0) = a, \quad y(L) = b.$$

The same solution is defined beyond the observed interval.

Tasks

1. Find the unique solution and justify uniqueness for every $L > 0$.
2. Prove that it is strictly positive throughout $[0, L]$ using the transformed quantity $u = e^t y$.
3. Determine exactly which endpoint pairs make the response nonnegative for every future time $t \geq 0$, not just between observations.
4. For $L = 1$, $a = 1$, $b = e^{-2}$, find the first zero after the observed interval and determine the sign for all later times. Explain why positive endpoint observations did not guarantee permanent positivity.

Question 9 – Solution

Strategy. The exponential transformation turns the repeated-root endpoint problem into linear interpolation, but its continuation beyond the interval is extrapolation.

Step 1: Solve the endpoint equations. Write $y = (A + Bt)e^{-t}$. The first datum sets $A = a$, and the second gives $a + BL = be^L$. Thus

$$y(t) = \left[a + \frac{be^L - a}{L}t \right] e^{-t}.$$

The two constants are uniquely determined because $L \neq 0$. This also proves existence by substitution into the repeated-root family.

Step 2: Prove positivity between the data points. For $0 \leq t \leq L$,

$$u(t) = e^t y(t) = (1 - t/L)a + (t/L)be^L.$$

The weights are nonnegative and sum to 1, while both endpoint values are positive. Hence $u > 0$ and therefore $y > 0$ throughout the closed interval.

Step 3: Determine when positivity persists. The affine function $u = a + Bt$ starts positive. It remains nonnegative for all $t \geq 0$ exactly when $B \geq 0$. Equivalently,

$$b \geq ae^{-L}.$$

At equality, u is constant and $y = ae^{-t}$. If $b < ae^{-L}$, the slope of u is negative and its eventual zero is unavoidable despite both observed values being positive.

Step 4: Check the concrete continuation. Here $B = e^{-1} - 1 < 0$, so

$$y(t) = [1 - (1 - e^{-1})t]e^{-t}.$$

Its unique zero is

$$t_z = \frac{1}{1 - e^{-1}} = \frac{e}{e - 1} > 1.$$

The response is negative for every $t > t_z$ and tends to zero from below. The given endpoint at 1 is indeed $[e^{-1}]e^{-1} = e^{-2}$. Linear interpolation of positive transformed data stays positive between them; extending that same negative-slope line beyond the interval eventually crosses zero. The endpoint data determine the continuation uniquely, but do not force it to preserve the observed sign forever.