

Question 8

For the dimensionless equation

$$y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1,$$

define the nonnegative quantity $E(t) = ((y'(t))^2 + (y(t))^2)/2$.

Tasks

1. Solve the IVP and find the time and height of the displacement maximum.
2. Differentiate E using the equation. Prove that E is strictly decreasing between any two distinct nonnegative times, even though its derivative can vanish at an isolated time.
3. Compute $E(0)$ and its value at the displacement maximum. Sketch y and E , and explain why increasing displacement is consistent with decreasing E .
4. Use the energy identity to calculate $\int_0^\infty (y')^2 dt$, and confirm the answer by integrating the explicit derivative.

Question 8 – Solution

Strategy. Distinguish a coordinate's amplitude from a quantity combining position and derivative, then integrate the exact dissipation identity.

Step 1: Solve and locate the transient maximum. The repeated root is -1 . The data give $y = (1 + 2t)e^{-t}$ and $y' = (1 - 2t)e^{-t}$. Hence the unique displacement maximum is

$$t_m = 1/2, \quad y(t_m) = 2e^{-1/2} > 1.$$

Step 2: Derive and interpret dissipation. Differentiating E and substituting $y'' = -2y' - y$ gives

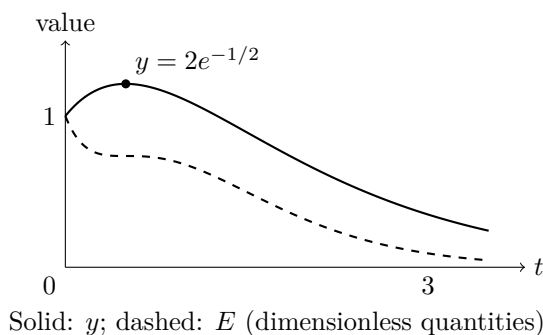
$$E' = y'y'' + yy' = y'(y'' + y) = -2(y')^2 \leq 0.$$

The derivative vanishes only at $t = 1/2$. On any interval $[s, t]$ with $s < t$, $(y')^2$ is positive except possibly at that one point, so its integral is strictly positive. Thus $E(t) - E(s) = -2 \int_s^t (y')^2 d\tau < 0$, proving strict decrease between distinct times.

Step 3: Compare the two quantities. Initially $E(0) = 1$. At the displacement peak, $y' = 0$, so

$$E(1/2) = \frac{1}{2}(2e^{-1/2})^2 = 2/e < 1.$$

The explicit expression is $E = (1 + 4t^2)e^{-2t}$.



Displacement initially increases, but the derivative contribution to E decreases enough for their sum to fall. The two curves represent the defined dimensionless quantities, not two competing claims about the same amplitude. Negative repeated roots ensure eventual decay without prohibiting an initial increase in y .

Step 4: Check the total integral in two ways. Since $y, y' \rightarrow 0$, also $E \rightarrow 0$. Integrating the dissipation identity gives $0 - 1 = -2 \int_0^\infty (y')^2 dt$, so the integral is $1/2$.

Directly, $(y')^2 = (1 - 4t + 4t^2)e^{-2t}$. Integration by parts gives the moments $\int_0^\infty e^{-2t} dt = 1/2$, $\int_0^\infty te^{-2t} dt = 1/4$ and $\int_0^\infty t^2 e^{-2t} dt = 1/4$. The resulting value $1/2 - 4(1/4) + 4(1/4) = 1/2$ agrees.