

Question 7

Consider the parameter family

$$y'' + 2y' + (1 - \mu)y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad -1 < \mu < 1.$$

For convenience, $\sinh z = (e^z - e^{-z})/2$. You may use $\sin z/z \rightarrow 1$ and $\sinh z/z \rightarrow 1$ as $z \rightarrow 0$.

Tasks

1. Solve the IVP separately for $\mu > 0$, $\mu = 0$ and $\mu < 0$, identifying the root type in each case.
2. Prove that the formulas from both sides tend to the repeated-root solution for each fixed $t \geq 0$ as $\mu \rightarrow 0$.
3. Determine which of these solutions have positive-time zeros. For $\mu < 0$, find the first positive zero and its limit as $\mu \uparrow 0$.
4. Prove that every solution of every equation in the stated parameter range tends to zero as $t \rightarrow \infty$. Explain why convergence of the displayed IVP solutions on any fixed time window does not preserve their number of positive-time zeros.

Question 7 – Solution

Strategy. Center the characteristic equation at its repeated root and compare the resulting real, repeated and complex formulas without conflating their global zero patterns.

Step 1: Solve each root case. The characteristic polynomial is $(r+1)^2 - \mu$. If $\mu > 0$, set $s = \sqrt{\mu} \in (0, 1)$. The roots are $-1 \pm s$ and the data give

$$y_\mu = e^{-t} \frac{\sinh(st)}{s}.$$

At $\mu = 0$, the root is -1 twice, and the solution is $y_0 = te^{-t}$. If $\mu < 0$, set $b = \sqrt{-\mu} > 0$; the roots are $-1 \pm ib$, giving

$$y_\mu = e^{-t} \frac{\sin(bt)}{b}.$$

Each formula has value 0 and derivative 1 at zero and solves its stated equation.

Step 2: Recover the repeated-root limit from either side. For $t > 0$, factor the first formula as $te^{-t} \sinh(st)/(st)$ and the third as $te^{-t} \sin(bt)/(bt)$. The supplied limits yield $y_\mu(t) \rightarrow te^{-t}$ from either side. At $t = 0$, all values are zero directly. Thus the repeated-root solution is the common fixed-time limit, rather than a missing special value.

Step 3: Locate zeros before taking their limit. For $\mu \geq 0$, the displayed solution is strictly positive for every $t > 0$, since $t > 0$ and $\sinh(st) > 0$. For $\mu < 0$, its positive zeros are

$$t_n = \frac{n\pi}{\sqrt{-\mu}}, \quad n = 1, 2, \dots$$

In particular, $t_1 \rightarrow \infty$ as $\mu \uparrow 0$. The infinitely many oscillations move beyond every fixed finite observation window.

Step 4: Separate decay from zero counts. For $0 < \mu < 1$, both roots $-1 \pm \sqrt{\mu}$ are negative; at zero, both repeated-root modes decay; for $\mu < 0$, the exponential factor e^{-t} bounds every fixed sine/cosine combination. Hence every solution decays throughout the stated parameter range.

On any fixed $[0, T]$, the ratios in Step 2 converge uniformly because their arguments lie in intervals shrinking to zero and both ratios extend continuously with value 1 there. Yet for every negative μ there are infinitely many later zeros, while y_0 has none for $t > 0$. Convergence on fixed windows does not control zeros that escape to infinity.