

Question 6

The repeated-root response $y(t) = te^{-t}$ solves $y'' + 2y' + y = 0$ with $y(0) = 0$, $y'(0) = 1$. Consider exponential bounds on $t \geq 0$.

Tasks

1. Find the maximum of y and verify that it tends to zero.
2. Determine whether any finite constant C can satisfy $y(t) \leq Ce^{-t}$ for every $t \geq 0$.
3. For a fixed $0 < \varepsilon < 1$, find the smallest C_ε such that $y(t) \leq C_\varepsilon e^{-(1-\varepsilon)t}$ for every $t \geq 0$. Prove optimality and sketch the bound for $\varepsilon = 1/2$ with its contact point.
4. Extend the argument to $w(t) = (A + Bt)e^{-t}$ by giving an explicit finite bound with rate $1 - \varepsilon$. Explain why eventual decay does not automatically give a bound with the exact exponent 1.

Question 6 – Solution

Strategy. Divide by the proposed exponential bound and maximize the resulting ratio, rather than judging only the root's sign.

Step 1: Find the transient peak. We have $y' = (1 - t)e^{-t}$, so the response rises until $t = 1$ and falls thereafter. Hence

$$\max_{t \geq 0} y(t) = 1/e.$$

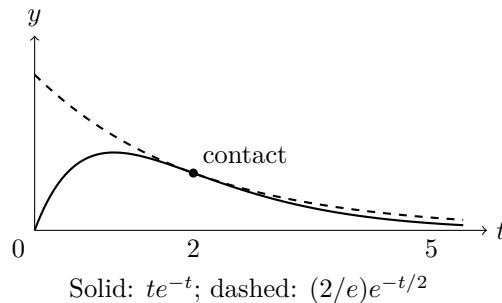
Also $t/e^t \rightarrow 0$, for example by l'Hôpital's rule, proving eventual decay.

Step 2: Test the exact root exponent. The proposed bound is equivalent to $t \leq C$ for all $t \geq 0$, which no finite C can satisfy. The exponential decay is modified by an unbounded polynomial prefactor.

Step 3: Find the sharp slightly slower bound. Dividing by $e^{-(1-\varepsilon)t}$ gives the ratio $te^{-\varepsilon t}$. Its derivative is $e^{-\varepsilon t}(1 - \varepsilon t)$, so its unique maximum occurs at $t = 1/\varepsilon$ and equals $1/(e\varepsilon)$. Thus

$$C_\varepsilon = \frac{1}{e\varepsilon}.$$

This constant is sufficient by the maximum calculation and necessary because equality holds at that time.



For $\varepsilon = 1/2$, the bound is $(2/e)e^{-t/2}$ and contact occurs at $t = 2$, where both values are $2e^{-2}$. Both derivatives equal $-e^{-2}$, confirming tangency rather than a crossing.

Step 4: Bound every fixed repeated-root solution. The triangle inequality gives

$$|w(t)|e^{(1-\varepsilon)t} \leq |A|e^{-\varepsilon t} + |B|te^{-\varepsilon t} \leq |A| + \frac{|B|}{e\varepsilon}.$$

Therefore $|w(t)| \leq (|A| + |B|/(e\varepsilon))e^{-(1-\varepsilon)t}$. This displayed constant need not be optimal for arbitrary A, B . A bound with the exact exponent 1 exists precisely when $B = 0$; for $B \neq 0$, the ratio $|A + Bt|$ is unbounded even though w itself decays.