

Question 5

For a real parameter r , consider

$$y'' - 2ry' + r^2y = 0.$$

A solution is forward bounded if bounded on $[0, \infty)$ and two-sided bounded if bounded on all of $\mathbb{R}$.

Tasks

1. Give the complete solution family for every real r , including $r = 0$.
2. Classify exactly which solutions are forward bounded and which tend to zero as $t \rightarrow \infty$, for each sign of r .
3. Classify all two-sided bounded solutions. Justify why a polynomial factor cannot cancel exponential growth in the growing time direction.
4. Explain why a repeated zero root behaves differently from a single zero root paired with a negative root. Give a counterexample to the statement that nonpositive characteristic roots always make every solution bounded.

Question 5 – Solution

Strategy. Treat the zero root separately and distinguish decay in one time direction from boundedness in both directions.

Step 1: Retain both repeated-root freedoms. For every r , the family is $y = (A + Bt)e^{rt}$. At $r = 0$ this becomes the affine family $y = A + Bt$ solving $y'' = 0$, not merely the constant family.

Step 2: Classify forward behavior. For $r < 0$, both e^{rt} and te^{rt} tend to zero, so every solution decays and is forward bounded. For $r = 0$, boundedness requires $B = 0$, while decay to zero requires $A = B = 0$. For $r > 0$, only the zero solution is forward bounded or tends to zero.

r	forward-bounded solutions	solutions tending to zero
$r < 0$	all solutions	all solutions
$r = 0$	$y = A$	$y = 0$
$r > 0$	$y = 0$	$y = 0$

For negative r , boundedness follows from the finite limit and continuity on every finite initial interval.

Step 3: Inspect the other time direction. If $B \neq 0$, the magnitude of $A + Bt$ is at least $|B||t|/2$ for sufficiently large $|t|$. If $B = 0$ but $A \neq 0$, it is a nonzero constant. Thus neither nonzero case can cancel the exponential in its growing direction.

When $r > 0$, that direction is $t \rightarrow \infty$; when $r < 0$, it is $t \rightarrow -\infty$. Hence the complete two-sided classification is

$$\boxed{r \neq 0 : y = 0 \text{ only}}, \quad \boxed{r = 0 : y = A \text{ for any constant } A}.$$

Step 4: Explain the exceptional repeated zero. A single zero root paired with a negative root gives a constant mode plus a decaying exponential, so every solution is forward bounded. A repeated zero root additionally gives the unbounded mode t .

For example, $y(t) = t$ solves $y'' = 0$, whose roots are both zero and therefore nonpositive, but is unbounded forward. Root signs alone are insufficient at a repeated zero; the polynomial multiplier must be included. For a strictly negative repeated root, the polynomial affects the rate and shape but not eventual decay.