

Question 3

For $\lambda > 0$ and $a > 0$, consider

$$y'' + 2\lambda y' + \lambda^2 y = 0, \quad y(0) = a, \quad y'(0) = b.$$

Classify all real slopes b on $t \geq 0$.

Tasks

1. Find the solution in terms of a, b, λ .
2. Prove exactly when the response remains nonnegative for every $t \geq 0$.
3. Prove exactly when it is both nonnegative and nonincreasing, treating the boundary slopes separately.
4. For the remaining slopes, find the unique zero or positive-time maximum, as appropriate. Summarize the cases on a diagram of $b/(\lambda a)$ and explain how the repeated root enters the classification.

Question 3 – Solution

Strategy. Reduce both sign and monotonicity to affine functions of time after factoring out a positive exponential.

Step 1: Resolve the data. The root is $-\lambda$ twice. Writing $y = (A + Bt)e^{-\lambda t}$ gives $A = a$, $B = b + \lambda a$. Thus

$$y = (a + (b + \lambda a)t)e^{-\lambda t}.$$

For brevity let $B = b + \lambda a$.

Step 2: Classify nonnegativity. The multiplier starts at $a > 0$. It stays positive at every finite nonnegative time exactly when $B \geq 0$; if $B < 0$, it crosses zero and becomes negative. Therefore

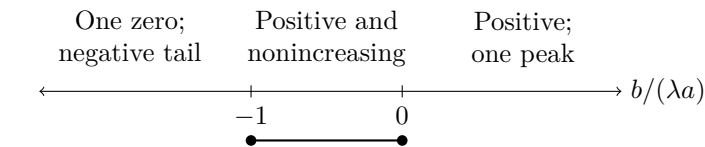
$$y(t) \geq 0 \text{ for all } t \geq 0 \iff b \geq -\lambda a.$$

At equality, the polynomial factor is constant and $y = ae^{-\lambda t}$.

Step 3: Impose nonincrease as well. Differentiation gives $y' = e^{-\lambda t}(b - \lambda Bt)$. With $B \geq 0$, this derivative is nonpositive for all $t \geq 0$ exactly when $b \leq 0$. Hence

$$-\lambda a \leq b \leq 0.$$

At $b = -\lambda a$, the response is a decreasing pure exponential. At $b = 0$, it has zero initial slope but negative derivative for every $t > 0$ because $B = \lambda a > 0$.



Both boundary slopes are included in the middle case.

Step 4: Classify the exterior ranges. If $b < -\lambda a$, then $B < 0$, and the unique zero is

$$t_z = -\frac{a}{b + \lambda a} > 0.$$

The response crosses from positive to negative and tends to zero from below. If $b > 0$, then $B > 0$ and the unique maximum occurs at

$$t_m = \frac{b}{\lambda(b + \lambda a)} > 0.$$

Here y' changes from positive to negative, and the response remains positive. The repeated root produces an affine multiplier; the zeros of that multiplier and its differentiated counterpart govern these thresholds. All cases still decay to zero.