

Question 9

For the decaying oscillation $y(t) = e^{-t} \sin t$, define its signed total area and its total absolute area by

$$I = \int_0^{\infty} y(t) dt, \quad J = \int_0^{\infty} |y(t)| dt.$$

Tasks

1. Find the corresponding monic homogeneous equation and initial data. State the signs on consecutive half-cycles.
2. Find an antiderivative of $e^{-t} \sin t$ and calculate I .
3. Calculate J exactly by comparing the absolute areas of successive half-cycles. Explain why $J > I$ even though both are finite.
4. Sketch the first two half-cycles with their signed areas indicated. Bound the absolute tail after an arbitrary $T \geq 0$, and give its exact value when $T = n\pi$ for a nonnegative integer n .

Question 9 – Solution

Strategy. Separate cancellation between lobes from decay of each lobe, then sum the absolute areas as a geometric series.

Step 1: Identify the equation and signs. The roots are $-1 \pm i$, so $y'' + 2y' + 2y = 0$. The derivative is $e^{-t}(\cos t - \sin t)$, giving $y(0) = 0$, $y'(0) = 1$. On $(n\pi, (n+1)\pi)$, the sign is $(-1)^n$: positive first, negative next, and alternating thereafter.

Step 2: Integrate the signed response. Integration by parts twice, or differentiation of the result, gives

$$F(t) = -\frac{1}{2}e^{-t}(\sin t + \cos t), \quad F' = e^{-t} \sin t.$$

Since $F(t) \rightarrow 0$ as $t \rightarrow \infty$ and $F(0) = -1/2$,

$$I = 1/2.$$

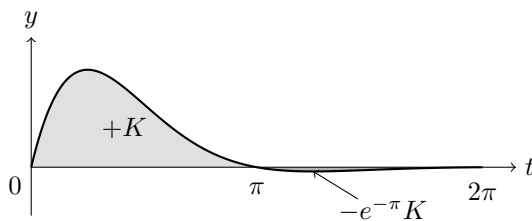
Step 3: Sum the magnitudes without cancellation. The first positive lobe has area

$$K = F(\pi) - F(0) = \frac{1 + e^{-\pi}}{2}.$$

The substitution $t = n\pi + s$ shows that the absolute area of the n th lobe is $e^{-n\pi}K$. Thus the convergent geometric sum gives

$$J = \frac{K}{1 - e^{-\pi}} = \frac{1 + e^{-\pi}}{2(1 - e^{-\pi})} > \frac{1}{2}.$$

The negative lobes reduce I but contribute positively to J . Infinitely many oscillations do not prevent absolute convergence because the lobe magnitudes decrease geometrically.



Signed lobe areas; both magnitudes contribute to J .

Step 4: Control the unobserved tail. Since $|\sin t| \leq 1$,

$$\int_T^\infty |y(t)| dt \leq \int_T^\infty e^{-t} dt = e^{-T}.$$

At a half-cycle boundary, substitution by $n\pi$ preserves $|\sin|$, so the exact absolute tail is $e^{-n\pi}J$. The signed area of the second plotted lobe is $-e^{-\pi}K$, while its contribution to J is $+e^{-\pi}K$.