

Question 8

Consider the endpoint problem

$$y'' + 2y' + 5y = 0, \quad y(0) = A, \quad y(L) = B, \quad L > 0.$$

The numbers A, B are endpoint values, not mode coefficients.

Tasks

1. Derive the endpoint equation for the remaining sine coefficient. Determine all lengths L for which every pair of endpoint values gives a unique solution.
2. At each excluded length, give the exact compatibility condition for existence and classify the number of solutions.
3. For $A = 0, B = 1$, solve when $L = \pi/4$ and determine what happens at $L = \pi/2$.
4. Keep $A = 0, B = 1$ but take $L = \pi/2 + \varepsilon$, where $0 < \varepsilon < \pi/4$. Find $y(\pi/4)$ and its leading behavior as $\varepsilon \downarrow 0$. Explain how unique solvability for every such ε can coexist with arbitrarily large interior values.

Question 8 – Solution

Strategy. An exponential prefactor never vanishes, so solvability and sensitivity are governed by the sine factor in the endpoint data map.

Step 1: Reduce to one coefficient. The roots are $-1 \pm 2i$. The first endpoint fixes the cosine coefficient, giving $y = e^{-t}(A \cos 2t + C \sin 2t)$. The second requires

$$C \sin 2L = Be^L - A \cos 2L.$$

If $\sin 2L \neq 0$, there is exactly one C , so unique solvability for arbitrary data holds exactly when

$$L \neq n\pi/2 \quad (n = 1, 2, 3, \dots).$$

Step 2: Classify the exceptional lengths. At $L = n\pi/2$, the condition becomes $0 = Be^L - (-1)^n A$. Thus

$$B = (-1)^n e^{-L} A$$

is necessary and sufficient for existence. If it holds, every real C works, giving infinitely many solutions. If it fails, there is no solution. The initial-value uniqueness theorem does not promise endpoint uniqueness at these lengths.

Step 3: Apply the two specified lengths. For $A = 0$, $B = 1$, $L = \pi/4$, we have $C = e^{\pi/4}$ and

$$y = e^{\pi/4-t} \sin 2t.$$

At $L = \pi/2$, compatibility would require $B = 0$, contradicting $B = 1$, so no solution exists.

Step 4: Quantify sensitivity near an exceptional length. For $L = \pi/2 + \varepsilon$, $\sin 2L = -\sin 2\varepsilon \neq 0$. Hence

$$y(t) = -\frac{e^{L-t} \sin 2t}{\sin 2\varepsilon}, \quad y(\pi/4) = -\frac{e^{\pi/4+\varepsilon}}{\sin 2\varepsilon}.$$

Using $\sin 2\varepsilon \sim 2\varepsilon$ and $e^\varepsilon \rightarrow 1$ gives

$$y(\pi/4) \sim -\frac{e^{\pi/4}}{2\varepsilon} \longrightarrow -\infty.$$

For each positive ε there is one finite coefficient, but the denominator approaches zero as the length approaches the incompatible endpoint problem. Uniqueness at each parameter value is not a uniform bound on the dependence on that parameter.