

Question 7

An unknown frequency $\beta > 0$ appears in

$$y'' + \beta^2 y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Exact observations at all nonnegative integer times report $y(n) = 0$.

Tasks

1. Solve the IVP in terms of β and determine every frequency compatible with the integer observations.
2. Explain why the observations, even with the known initial slope, do not imply the zero solution and do not identify a unique frequency.
3. An additional exact observation gives $y(1/2) = 1/\pi$. Determine which of the previously possible frequencies remain. Prove that no larger candidate frequency can fit it.
4. Sketch the candidates with $\beta = \pi$ and $\beta = 3\pi$ on $[0, 2]$, marking the integer sampling times. Explain what feature of the observation schedule hides their different oscillations.

Question 7 – Solution

Strategy. Solve before interpreting the data; a sampling grid can repeatedly land on zeros of a nonzero oscillation.

Step 1: Determine all admissible frequencies. The roots are $\pm i\beta$, and the data give

$$y(t) = \frac{\sin \beta t}{\beta}.$$

In particular, $y(1) = 0$ implies $\sin \beta = 0$, so $\beta = m\pi$ for a positive integer m . Conversely, each such frequency makes every integer observation zero. The complete candidate set is therefore

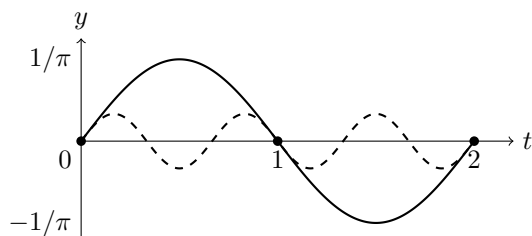
$$\beta = m\pi, \quad m = 1, 2, 3, \dots$$

Step 2: Retain the slope information correctly. Each candidate has $y'(t) = \cos(m\pi t)$ and hence $y'(0) = 1$, so each is nonzero. The zero function would violate that slope. The slope sets the sine coefficient to $1/(m\pi)$, but still allows infinitely many frequencies. A known initial slope does not rescue a schedule that misses all intermediate oscillations.

Step 3: Use the extra observation to select a frequency. For the candidate indexed by m ,

$$y(1/2) = \frac{\sin(m\pi/2)}{m\pi}.$$

For $m = 1$ this equals $1/\pi$. For every $m > 1$, its absolute value is at most $1/(m\pi) < 1/\pi$, making agreement impossible. Thus the new datum selects exactly $\beta = \pi$ and $y = \sin(\pi t)/\pi$.



Solid: $\beta = \pi$; dashed: $\beta = 3\pi$; dots: samples

Step 4: Explain the indistinguishable samples. The integer grid samples an integer number of half-cycles for every candidate, so it always observes a zero. The two plotted curves differ between those times, including in amplitude, yet share all grid values and the initial slope. This is an explicit sampling ambiguity, not a violation of uniqueness: each fixed equation has one solution for its data, while different frequencies define different equations.