

Question 6

For constants $a \in \mathbb{R}$, $b > 0$, consider

$$y'' + 2ay' + (a^2 + b^2)y = 0.$$

Define transformed coordinates

$$u = e^{at}y, \quad v = \frac{e^{at}(y' + ay)}{b}.$$

The axes in a plot of (u, v) are these transformed quantities, not time and displacement.

Tasks

1. Differentiate u, v and prove $u' = bv$, $v' = -bu$.
2. Deduce a conserved quantity and express its value using $y(0) = Y$, $y'(0) = V$. Relate it to the amplitude of the trigonometric factor in y .
3. For $a = 1$, $b = 2$, $Y = 1$, $V = -1$, find y, u, v . Sketch the transformed curve with its initial point, direction and quarter-cycle points.
4. Explain why the circle in these transformed coordinates does not imply constant amplitude of the original response. Determine exactly which original initial data make the conserved quantity zero and what solution follows.

Question 6 – Solution

Strategy. Remove the exponential factor and scale the remaining derivative so the two trigonometric coordinates have equal amplitudes.

Step 1: Derive the transformed equations. Directly, $u' = e^{at}(y' + ay) = bv$. Also,

$$v' = \frac{e^{at}}{b}(y'' + 2ay' + a^2y) = \frac{e^{at}}{b}(-b^2y) = -bu.$$

The original equation is essential in the second equality.

Step 2: Identify the invariant and amplitude. Differentiating gives $(u^2 + v^2)' = 2u(bv) + 2v(-bu) = 0$. At zero,

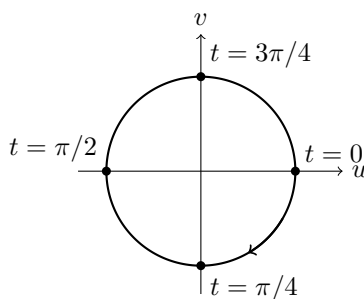
$$u^2 + v^2 = Y^2 + \frac{(V + aY)^2}{b^2} = R^2.$$

The real solution has trigonometric coefficients $A = Y$, $B = (V + aY)/b$, so $R = \sqrt{A^2 + B^2}$ is exactly its trigonometric amplitude.

Step 3: Trace the concrete circle. For the stated data, $A = 1$, $B = 0$. Thus

$$y = e^{-t} \cos 2t, \quad u = \cos 2t, \quad v = -\sin 2t.$$

The initial point is $(1, 0)$ and its velocity is $(0, -2)$, so motion is clockwise. At times $0, \pi/4, \pi/2, 3\pi/4$ the points are $(1, 0), (0, -1), (-1, 0), (0, 1)$, respectively; the cycle closes at $t = \pi$.



$$u^2 + v^2 = 1; \text{ equal axis scales}$$

Step 4: Translate back to the original quantity. The transformation removed e^{-at} : the original response satisfies $|y| \leq Re^{-at}$. Its amplitude decays for $a > 0$, is constant for $a = 0$, and grows for $a < 0$. A circle in (u, v) describes a normalized oscillation, not constant original displacement amplitude or a claim about physical energy.

Because $b > 0$, the sum of squares R^2 is zero exactly when $Y = 0$ and $V + aY = 0$, hence exactly when $Y = V = 0$. These data give $y \equiv 0$, consistent with initial-value uniqueness.