

Question 5

For a real parameter a , consider

$$y'' + 2ay' + (a^2 + 9)y = 0.$$

All qualitative claims concern $t \geq 0$. A solution is nontrivial if it is not identically zero.

Tasks

1. Find the roots and the real general solution for every a .
2. Classify when every solution tends to zero and when every solution is bounded. Describe the nontrivial behavior at $a = 0$.
3. Prove that every nontrivial solution has infinitely many simple zeros, with the same spacing for all a . Can any such solution be eventually of one strict sign?
4. For $a < 0$, distinguish unbounded magnitude from the assertion $|y(t)| \rightarrow \infty$. Give explicit sequences of times that prove the correct conclusions.

Question 5 – Solution

Strategy. Separate the exponential amplitude from the trigonometric zero pattern; growth of the former does not erase zeros of the latter.

Step 1: Find the universal complex-root form. The polynomial is $(r + a)^2 + 9$, so the roots are $-a \pm 3i$. Hence

$$y = e^{-at}(A \cos 3t + B \sin 3t).$$

For a nontrivial solution, $R = \sqrt{A^2 + B^2} > 0$ and some phase δ give $y = Re^{-at} \cos(3t - \delta)$.

Step 2: Classify decay and boundedness. If $a > 0$, $|y| \leq Re^{-at} \rightarrow 0$. At $a = 0$, all solutions are bounded, and every nontrivial one is periodic with least positive period $2\pi/3$ and does not tend to zero. If $a < 0$, the magnitudes at trigonometric peaks grow without bound. Therefore

$$\boxed{\text{all solutions decay} \iff a > 0}, \quad \boxed{\text{all solutions are bounded} \iff a \geq 0}.$$

The zero solution is bounded and decays for every parameter value.

Step 3: Track the zeros and their slopes. Zeros occur when $3t - \delta = \pi/2 + n\pi$, so

$$t_n = \frac{\delta + \pi/2 + n\pi}{3},$$

with all integers n giving nonnegative times retained. There are infinitely many, spaced by $\pi/3$. At each zero,

$$y'(t_n) = -3Re^{-at_n} \sin(\pi/2 + n\pi) \neq 0.$$

Thus every zero is simple and the sign changes there. No nontrivial solution is eventually strictly positive or strictly negative, even when its amplitude decays.

Step 4: State growth without an incorrect limit. For $a < 0$, choose large integers n and set $s_n = (\delta + 2n\pi)/3$. Then $y(s_n) = Re^{-as_n} \rightarrow +\infty$. At $u_n = (\delta + (2n + 1)\pi)/3$, $y(u_n) = -Re^{-au_n} \rightarrow -\infty$. But along the zero sequence, $|y(t_n)| = 0$.

Consequently the solution has unbounded magnitude, $\limsup y = +\infty$ and $\liminf y = -\infty$, while $\liminf |y| = 0$. The limit $|y(t)| \rightarrow \infty$ is false. A growing envelope describes arbitrarily large excursions, not a lower bound on the response at every late time.