

Question 4

A nonzero solution of an unknown monic equation $y'' + py' + qy = 0$ has characteristic roots $\alpha \pm i\beta$, where $\beta > 0$. Two consecutive positive local maxima occur at $t = 1$ and $t = 3$, with respective heights 4 and 1. The equation has real constant coefficients, and there is no forcing.

Tasks

1. Determine β and α from the peak times and heights. Justify why positive maxima are separated by a full cycle even when $\alpha \neq 0$.
2. Recover p, q and write the equation exactly.
3. Reconstruct the unique solution from the peak data at $t = 1$. Verify that the point at $t = 3$ is also a positive local maximum of height 1.
4. If the word consecutive were removed, would the two peak observations still uniquely determine the equation? Describe the resulting possibilities for β and explain the ambiguity.

Question 4 – Solution

Strategy. Use the interval between like-signed extrema for frequency and their height ratio for the real part of the roots.

Step 1: Extract frequency and decay. Write $y = Re^{\alpha t} \cos(\beta t - \delta)$. Stationarity requires $\alpha \cos(\beta t - \delta) - \beta \sin(\beta t - \delta) = 0$. Successive extrema have phase separation π and alternate sign, so consecutive positive maxima are separated by $2\pi/\beta$.

The observed separation is 2, giving $\beta = \pi$. The trigonometric factors at those maxima agree, so $1/4 = e^{2\alpha}$ and

$$\alpha = -\ln 2.$$

Step 2: Recover the real coefficients. Let $k = \ln 2 > 0$. The characteristic polynomial is $(r + k)^2 + \pi^2$, hence

$$y'' + 2ky' + (k^2 + \pi^2)y = 0.$$

Step 3: Reconstruct from value and zero slope. Using $x = t - 1$, write $y = e^{-kx}(A \cos \pi x + B \sin \pi x)$. The peak gives $A = 4$ and $-kA + \pi B = 0$, so

$$y = e^{-k(t-1)} \left[4 \cos(\pi(t-1)) + \frac{4k}{\pi} \sin(\pi(t-1)) \right].$$

At $t = 3$, $x = 2$, and $y = 4e^{-2k} = 1$, $y' = 0$. At either point the equation gives $y'' = -(k^2 + \pi^2)y < 0$, verifying strict positive local maxima. The solution is unique because its value and slope at 1 are specified for a regular linear equation.

Step 4: Identify what consecutiveness contributes. Without that condition, the time interval 2 could contain m full cycles, where m is any positive integer. Then $\beta = m\pi$ while the height ratio still fixes $\alpha = -\ln 2$. Each choice gives a different equation and a solution with the stated peaks, but for $m > 1$ there are additional positive maxima between them. The observed heights alone do not resolve the missing cycle count.