

Question 3

For the response $y(t) = e^{-t} \cos 2t$ on $t \geq 0$, the curves $y = \pm e^{-t}$ are called its exponential envelopes because $|y(t)| \leq e^{-t}$.

Tasks

1. Find a monic homogeneous equation with this solution and verify its initial value and slope.
2. Find all positive-time stationary points, classify them, and identify the first minimum and the first positive-time maximum.
3. Determine where the response touches an envelope. Are these times stationary points? Find the ratio of each stationary-point magnitude to the envelope height there.
4. Sketch the response and envelopes on $[0, 2\pi]$, marking the first minimum. Determine the ratio of consecutive positive local maxima and explain why envelope contacts and extrema must be distinguished.

Question 3 – Solution

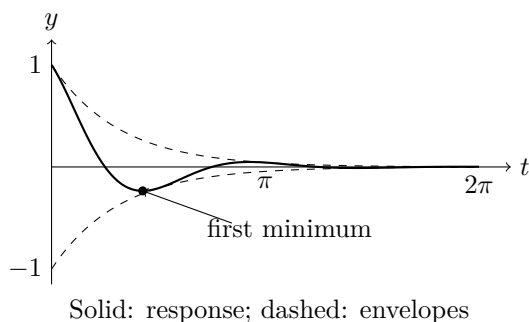
Strategy. Differentiate the entire product, including the exponential factor, before locating peaks.

Step 1: Verify the complex-root model. The roots $-1 \pm 2i$ give $y'' + 2y' + 5y = 0$. Differentiation yields $y' = e^{-t}(-\cos 2t - 2\sin 2t)$, so $y(0) = 1$ and $y'(0) = -1$. Substituting a second derivative verifies the equation.

Step 2: Locate and classify the extrema. Let $\theta = \arctan(1/2) \in (0, \pi/2)$. The stationary condition is $\cos 2t + 2\sin 2t = 0$, giving

$$t_k = (k\pi - \theta)/2, \quad k = 1, 2, \dots$$

At these times $\cos 2t_k = (-1)^k \cos \theta = (-1)^k 2/\sqrt{5}$. Since $y' = 0$, the ODE gives $y'' = -5y$. Thus odd k give negative minima and even k give positive maxima. The first minimum is at $(\pi - \theta)/2$; the first positive-time maximum is at $\pi - \theta/2$.



Step 3: Compare extrema with envelope contacts. Equality $|y| = e^{-t}$ requires $|\cos 2t| = 1$, or $t = n\pi/2$ for $n \geq 0$. At those times $y' = -y \neq 0$, so they are not stationary points. The response does share the tangent of the corresponding envelope there.

At every stationary point,

$$|y(t_k)| = \frac{2}{\sqrt{5}} e^{-t_k} < e^{-t_k}.$$

The decreasing envelope shifts the extrema away from cosine peaks.

Step 4: Compare like-signed peaks. Consecutive positive local maxima have indices k and $k + 2$, so their time separation is π . Their common trigonometric factor cancels, giving the height ratio $e^{-\pi}$. This is the envelope decay over a full cycle, even though the maxima occur strictly below the envelopes. At $t = 0$ the solution has a one-sided endpoint maximum; it is not a positive-time stationary maximum.