

Question 2

Consider

$$y'' + 2y' + 5y = 0, \quad y(0) = -1, \quad y'(0) = 3.$$

Use the phase convention $y = Re^{-t} \cos(2t - \delta)$ with $R > 0$ and $0 \leq \delta < 2\pi$.

Tasks

1. Solve the IVP in sine/cosine form.
2. Determine R, δ in the stated convention and verify the initial value and derivative in phase form.
3. Find every zero for $t \geq 0$, the sign immediately after the first zero, and the spacing between successive zeros.
4. Explain why solving only $\tan \delta = B/A$ can give the wrong phase quadrant. Diagnose the choice $\delta = -\pi/4$ with positive amplitude for this IVP.

Question 2 – Solution

Strategy. Determine the sine and cosine coefficients before choosing the phase quadrant; then use the phase to locate zeros.

Step 1: Solve the real IVP. The roots are $-1 \pm 2i$. For $y = e^{-t}(A \cos 2t + B \sin 2t)$, the data give $A = -1$ and $-A + 2B = 3$. Thus $B = 1$ and

$$y = e^{-t}(-\cos 2t + \sin 2t).$$

Step 2: Choose the correct quadrant. Expansion of $R \cos(2t - \delta)$ gives $A = R \cos \delta$, $B = R \sin \delta$. Therefore

$$R = \sqrt{A^2 + B^2} = \sqrt{2}, \quad \cos \delta = -1/\sqrt{2}, \quad \sin \delta = 1/\sqrt{2}.$$

The prescribed convention selects $\delta = 3\pi/4$, yielding

$$y = \sqrt{2}e^{-t} \cos(2t - 3\pi/4).$$

At zero the value is -1 , while the derivative is $\sqrt{2}[-\cos(-3\pi/4) - 2\sin(-3\pi/4)] = 3$, as required.

Step 3: Enumerate the nonnegative zeros. The exponential is never zero. The first zero occurs when $2t - 3\pi/4 = -\pi/2$, so all nonnegative zeros are

$$t_n = \frac{\pi}{8} + \frac{n\pi}{2}, \quad n = 0, 1, 2, \dots$$

At t_0 , the derivative is $2\sqrt{2}e^{-\pi/8} > 0$, so the solution crosses from negative to positive. Successive zeros are separated by $\pi/2$, half the trigonometric period π . The decaying solution itself is not periodic.

Step 4: Resolve the tangent ambiguity. The equation $\tan \delta = -1$ does not distinguish quadrants II and IV. With positive $R = \sqrt{2}$, the phase $-\pi/4$ gives $A = 1$, $B = -1$, which is the negative of the required solution. It has value 1 and slope -3 at zero. The signs of both sine and cosine, together with the amplitude convention, determine the phase; a tangent value alone does not.