

Question 1

Consider

$$2y'' + 4y' + 10y = 0, \quad y(1) = 2, \quad y'(1) = 2.$$

You may use Euler's identity $e^{i\theta} = \cos \theta + i \sin \theta$ and the continuous-coefficient linear initial-value uniqueness theorem.

Tasks

1. Find the characteristic roots. Use their complex exponential solutions to construct two real solutions.
2. Write a real solution family based at $t = 1$ and determine the coefficients from the initial data.
3. Verify the solution in the original equation and both data. Show that the real family can realize arbitrary initial value and slope at 1, and hence is complete.
4. In the complex form $c_+e^{(-1+2i)(t-1)} + c_-e^{(-1-2i)(t-1)}$, determine c_+, c_- for this IVP. Explain why taking both coefficients real and equal would lose some real solutions.

Question 1 – Solution

Strategy. Pair conjugate exponentials to obtain real functions, retaining two independent real data freedoms.

Step 1: Find the roots and real building blocks. The characteristic equation is $2r^2 + 4r + 10 = 0$, or $(r + 1)^2 + 4 = 0$, giving

$$r = -1 \pm 2i.$$

The half-sum and the difference divided by $2i$ of $e^{(-1 \pm 2i)t}$ give, respectively, $e^{-t} \cos 2t$ and $e^{-t} \sin 2t$. These solve the real equation because it is linear with real coefficients.

Step 2: Fit the shifted family. Set $x = t - 1$ and write $y = e^{-x}(A \cos 2x + B \sin 2x)$. At $x = 0$, $y = A$ and $y' = -A + 2B$. Thus $A = 2$ and $B = 2$, so

$$y = 2e^{-(t-1)}(\cos(2(t-1)) + \sin(2(t-1))).$$

Step 3: Verify the equation and completeness. Writing $y = e^{-x}u$ gives $y' = e^{-x}(u' - u)$ and $y'' = e^{-x}(u'' - 2u' + u)$. Hence

$$2y'' + 4y' + 10y = 2e^{-x}(u'' + 4u) = 0.$$

The value and slope at 1 are 2 and $-2 + 4 = 2$. More generally, data $y(1) = a$, $y'(1) = b$ give $A = a$, $B = (a + b)/2$, uniquely. Any solution shares its data with one family member; the linear theorem forces equality on $\mathbb{R}$.

Step 4: Interpret the complex constants correctly. The real combination satisfies

$$A \cos 2x + B \sin 2x = \frac{A - iB}{2} e^{2ix} + \frac{A + iB}{2} e^{-2ix}.$$

Thus $\boxed{c_+ = 1 - i, \quad c_- = 1 + i}$ for this IVP. The constants are conjugates, so the imaginary parts cancel. Equal real constants would force $B = 0$, permitting only initial slopes $b = -a$ in the shifted basis. Real solutions require conjugate coefficients, not necessarily real coefficients in the complex basis.