

Question 9

Let D denote differentiation with respect to t . Consider

$$y'' - 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = 4.$$

A student reasons that $(D - 3)(D - 1)y = 0$ means every solution must satisfy either $(D - 1)y = 0$ or $(D - 3)y = 0$.

Tasks

1. Verify the operator factorization by expanding it on a twice-differentiable function.
2. Set $z = y' - y$. Derive and solve its first-order initial-value problem.
3. Solve $y' - y = z$ using an integrating factor and the original value $y(0) = 2$. Verify the resulting solution and both initial data.
4. Test the student's either/or claim on your solution. Explain why a product of differential operators must not be interpreted as a product of two numbers acting separately on y .

Question 9 – Solution

Strategy. Factorization creates a sequence of two first-order equations, not a rule requiring one of two first-order residuals to vanish.

Step 1: Expand the operator composition. Applying $(D - 3)$ to $(D - 1)y = y' - y$ gives

$$(y' - y)' - 3(y' - y) = y'' - 4y' + 3y.$$

Thus the factorization is valid and its characteristic roots are 1 and 3.

Step 2: Solve for the intermediate quantity. With $z = y' - y$, the equation is $z' - 3z = 0$. The data imply $z(0) = 4 - 2 = 2$. Separating or using an integrating factor yields

$$\boxed{z = 2e^{3t}}.$$

The intermediate residual is nonzero, yet it is annihilated by $D - 3$.

Step 3: Recover the original solution. We must solve $y' - y = 2e^{3t}$. Multiplication by e^{-t} gives

$$(e^{-t}y)' = 2e^{2t}.$$

Integrating from 0 to t and using $y(0) = 2$ gives $e^{-t}y - 2 = e^{2t} - 1$. Hence

$$\boxed{y = e^t + e^{3t}}.$$

Indeed $y(0) = 2$, $y'(0) = 1 + 3 = 4$, and

$$y'' - 4y' + 3y = (1 - 4 + 3)e^t + (9 - 12 + 3)e^{3t} = 0.$$

The construction recovers both modes by first-order methods and gives the unique solution with the prescribed data.

Step 4: Refute the incorrect either/or inference. For this solution,

$$(D - 1)y = 2e^{3t} \neq 0, \quad (D - 3)y = -2e^t \neq 0.$$

Nevertheless, $(D - 3)((D - 1)y) = (D - 3)(2e^{3t}) = 0$. An operator composition first produces a new function and then applies another operator to it. A nonzero function can be sent to zero by differentiation combined with multiplication. The scalar zero-product rule therefore does not imply the proposed either/or claim. Sums of the two modes must be retained, not just the two separate pure-mode families.