

Question 8

The positive response

$$y(t) = e^{-t} + 9e^{-3t} \quad (t \geq 0)$$

solves a homogeneous second-order equation with real distinct roots. Define its instantaneous decay rate by $R(t) = -y'(t)/y(t)$.

Tasks

1. Recover the monic differential equation and verify the initial value and slope.
2. Express R as a weighted average of the two decay rates. Find $R(0)$, its limit, and the unique time when $R = 2$.
3. Differentiate R using the differential equation to prove that it is strictly decreasing. Sketch R with its limiting level marked.
4. A constant-rate extrapolation from the initial data predicts $\hat{y}(t) = 10e^{-(14/5)t}$. Prove whether this estimate lies above or below the actual response for every $t > 0$, using the curvature of $\ln y$.

Question 8 – Solution

Strategy. A mixture of two exponential modes need not have a constant effective decay rate; use its logarithmic derivative to quantify the change.

Step 1: Recover and check the equation. The roots are $-1, -3$, so the monic characteristic polynomial is $r^2 + 4r + 3$. Hence $y'' + 4y' + 3y = 0$. Direct differentiation gives $y' = -e^{-t} - 27e^{-3t}$, so

$$y(0) = 10, \quad y'(0) = -28.$$

The response is positive because both mode terms are positive.

Step 2: Read the rate as a changing weighted average. Let $z = 9e^{-2t} > 0$. Then

$$R = \frac{e^{-t} + 27e^{-3t}}{e^{-t} + 9e^{-3t}} = \frac{1 + 3z}{1 + z}.$$

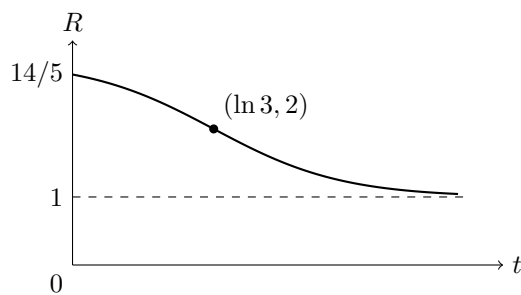
This averages 1 and 3 with positive weights 1, z , so $1 < R < 3$. In particular,

$$R(0) = 14/5, \quad R \rightarrow 1, \quad R = 2 \iff z = 1 \iff t = \ln 3.$$

Step 3: Derive the rate equation. Using $y'' = -4y' - 3y$ and $y'/y = -R$,

$$R' = -\frac{y''}{y} + (y'/y)^2 = R^2 - 4R + 3 = (R - 1)(R - 3) < 0.$$

Thus R is strictly decreasing; its long-time limit is the slower mode's rate, rather than an unchanging average of the roots.



The effective decay rate decreases toward 1.

Step 4: Compare with a constant-rate prediction. Set $f = \ln y$. Then $f' = -R$ and $f'' = -R' > 0$. Since f' is strictly increasing, integration on $[0, t]$ for $t > 0$ gives

$$f(t) > f(0) + tf'(0) = \ln 10 - \frac{14}{5}t.$$

Exponentiation preserves this inequality, so $y(t) > 10e^{-(14/5)t}$ for every $t > 0$. The extrapolation matches the initial value and slope but decays too quickly afterward. The changing mode weights, not an error in the initial data, cause the discrepancy.