

Question 7

For a real parameter α , consider

$$y'' + 2\alpha y' + (\alpha^2 - 4)y = 0.$$

A solution is forward bounded if it is bounded on $[0, \infty)$. Distinguish statements about every solution from statements about selected nonzero solutions.

Tasks

1. Find both roots and show that they are real and distinct for every real α .
2. Determine exactly when every solution tends to zero as $t \rightarrow \infty$.
3. Determine exactly when every solution is forward bounded. Treat $\alpha = 2$ explicitly.
4. For all $\alpha \leq 2$, classify the individual forward-bounded solutions, including $\alpha = -2$. Give a complete parameter table and explain why a zero root must not be treated as a negative root.

Question 7 – Solution

Strategy. Track the two root signs and retain the distinction between a mode being available and its coefficient actually being present.

Step 1: Keep the root separation visible. The characteristic polynomial is $(r + \alpha)^2 - 4$, so

$$\boxed{r_+ = 2 - \alpha, \quad r_- = -2 - \alpha}, \quad r_+ - r_- = 4.$$

Consequently $y = Ae^{(2-\alpha)t} + Be^{(-2-\alpha)t}$ for every real α ; no repeated or nonreal roots occur.

Step 2: Require both modes to decay. Every solution tends to zero exactly when both roots are negative. Since r_+ is the larger root, this is equivalent to

$$\boxed{\alpha > 2}.$$

If $\alpha \leq 2$, choosing $A = 1$, $B = 0$ gives a constant or growing mode, disproving universal decay.

Step 3: Allow a constant mode for boundedness. Every solution is forward bounded exactly when both roots are nonpositive, or $\boxed{\alpha \geq 2}$. At $\alpha = 2$,

$$y = A + Be^{-4t} \rightarrow A.$$

All solutions are bounded, but only those with $A = 0$ tend to zero. A zero root supplies a constant mode rather than a decaying one.

Step 4: Classify the selected bounded solutions. Using the coefficients A, B from Step 1, the full classification is

α	forward-bounded solutions	their limits
$\alpha < -2$	$y = 0$	0
$\alpha = -2$	$y = B \quad (A = 0)$	B
$-2 < \alpha < 2$	$y = Be^{(-2-\alpha)t} \quad (A = 0)$	0
$\alpha = 2$	$y = A + Be^{-4t}$	A
$\alpha > 2$	all solutions	0

For $\alpha < -2$, both roots are positive. If $A \neq 0$, the faster mode forces unbounded magnitude; if $A = 0$ and $B \neq 0$, the remaining positive mode still grows. Distinct growth rates cannot cancel for all large times.

At $\alpha = -2$ the roots are 4 and 0, so only constant solutions are bounded. For $-2 < \alpha < 2$, one positive mode must be removed, leaving a decaying mode. These boundary cases show why a classification based only on whether roots are real would be incomplete.