

Question 6

Consider homogeneous constant-coefficient equations with two distinct real roots $r_1 < r_2$ and endpoint data $y(0) = a$, $y(L) = b$, where $L > 0$.

Tasks

1. Prove that every pair $a, b \in \mathbb{R}$ determines exactly one solution by solving for the two mode coefficients. Identify the quantity that cannot vanish.
2. Solve $y'' - 3y' + 2y = 0$ with $y(0) = y(\ln 2) = 1$.
3. Prove that a nonzero combination $Ae^{r_1 t} + Be^{r_2 t}$ has at most one real zero. Treat vanishing coefficients explicitly, and deduce positivity between strictly positive endpoints.
4. For the solution in Task 2, find its maximum on $[0, \ln 2]$ and sketch it. Does positivity between positive endpoints mean the solution stays below its endpoint values?

Question 6 – Solution

Strategy. Use the strict inequality of real exponentials to solve the endpoint problem and control the number of zeros.

Step 1: Invert the endpoint constraints. For $y = Ae^{r_1 t} + Be^{r_2 t}$, the conditions give $A + B = a$ and $Ae^{r_1 L} + Be^{r_2 L} = b$. Therefore

$$B = \frac{b - ae^{r_1 L}}{e^{r_2 L} - e^{r_1 L}}, \quad A = a - B.$$

Because $r_2 > r_1$ and $L > 0$, the denominator is strictly positive. There is exactly one coefficient pair for all endpoint data, unlike some oscillatory endpoint problems.

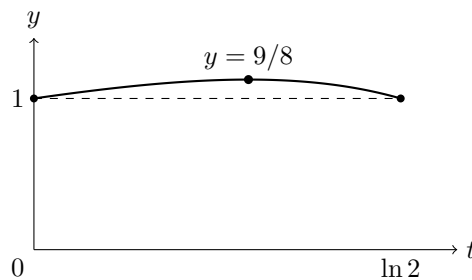
Step 2: Fit the concrete equation. Here the roots are 1 and 2. At $L = \ln 2$, the equations are $A + B = 1$ and $2A + 4B = 1$. Hence

$$y = \frac{3}{2}e^t - \frac{1}{2}e^{2t}.$$

Both endpoints have value 1, and each mode satisfies the original equation.

Step 3: Bound the number of zeros and deduce positivity. Factor the general solution as $e^{r_1 t}(A + Be^{(r_2 - r_1)t})$. If $B \neq 0$, the bracket is strictly monotone, so it has at most one zero. If $B = 0$, a nonzero solution is a nonvanishing pure exponential; the case $A = 0$ is likewise nonvanishing.

If a solution with positive endpoints became negative inside, continuity would force two distinct zeros. If it merely touched zero inside while staying nonnegative, that interior minimum would have $y = y' = 0$; initial-value uniqueness would force the identically zero solution, contradicting the endpoints. Thus it is strictly positive throughout the interval.



Positive endpoints do not bound the interior maximum.

Step 4: Distinguish positivity from an upper bound. For the concrete solution, $y' = e^t(3/2 - e^t)$ changes from positive to negative at $t = \ln(3/2)$, which lies between 0 and $\ln 2$. Its value there is

$$y_{\max} = \frac{3}{2} \frac{3}{2} - \frac{1}{2} \frac{9}{4} = \frac{9}{8} > 1.$$

The curve stays positive but exceeds both endpoint values. A restriction on zeros is not a maximum principle bounding the response by its endpoints.