

Question 4

For the equation

$$y'' + 3y' + 2y = 0, \quad y(0) = a > 0, \quad y'(0) = b,$$

classify the entire range of real initial slopes b on the time interval $t \geq 0$.

Tasks

1. Find the solution in terms of a, b .
2. Determine exactly when $y(t) \geq 0$ for every $t \geq 0$. Prove both necessity and sufficiency.
3. Determine exactly when the solution is both nonnegative and nonincreasing on $[0, \infty)$, including boundary slopes.
4. For the remaining slopes, find the unique zero when one occurs, or the unique positive-time maximum when one occurs. Summarize the cases on a diagram of the ratio b/a .

Question 4 – Solution

Strategy. Factor out a positive exponential and reduce sign questions to affine expressions in $z = e^{-t} \in (0, 1]$.

Step 1: Fit the real-root solution. The roots are $-1, -2$. Solving $A + B = a$, $-A - 2B = b$ gives

$$y = (2a + b)e^{-t} - (a + b)e^{-2t}.$$

Write $A = 2a + b$, $B = -(a + b)$, so $A + B = a$.

Step 2: Classify nonnegativity. Since $y = e^{-t}(A + Bz)$, its sign is the sign of an affine function with values A at $z = 0$ and $a > 0$ at $z = 1$. If $A \geq 0$, it is positive for all $0 < z \leq 1$. If $A < 0$, it is negative for sufficiently small positive z . Therefore

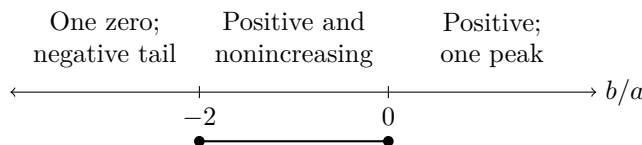
$$y(t) \geq 0 \text{ for all } t \geq 0 \iff b \geq -2a.$$

At $b = -2a$, the solution is the strictly positive ae^{-2t} at every finite time.

Step 3: Impose monotonicity as well. We have $y' = -e^{-t}(A + 2Bz)$. Its bracket has endpoint values A and $A + 2B = -b$. Thus it is nonnegative on $0 < z \leq 1$ exactly when $A \geq 0$ and $b \leq 0$. Combining the conditions yields

$$-2a \leq b \leq 0.$$

At $b = 0$, the derivative vanishes initially but is negative for every $t > 0$. Both boundary slopes are included.



Both boundary slopes belong to the middle case.

Step 4: Resolve the two remaining regimes. If $b < -2a$, solving $A + Be^{-t} = 0$ gives the unique zero

$$t_z = \ln \frac{a + b}{2a + b} > 0.$$

Both numerator and denominator are negative, and their ratio exceeds 1. The solution crosses from positive to negative and tends to zero from below.

If $b > 0$, it remains positive but initially rises. Solving $y' = 0$ gives the unique positive-time maximum at

$$t_m = \ln \frac{2(a + b)}{2a + b} > 0.$$

The derivative changes from positive to negative there. These cases exhaust all real b ; every solution still tends to zero because both roots are negative.