

Question 3

A transient response satisfies

$$y'' + 5y' + 6y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Only $t \geq 0$ is considered in the qualitative questions.

Tasks

1. Solve the initial-value problem using its real distinct roots.
2. Prove that $y(t) > 0$ for $t > 0$, even though $y(t) \rightarrow 0$ as $t \rightarrow \infty$.
3. Find the unique maximum, including its time and height, and sketch the response with that point marked.
4. Calculate $\int_0^\infty y(t) dt$ in two ways: from the explicit formula and by integrating the differential equation. Explain why negative roots alone do not imply monotone decay from the initial value.

Question 3 – Solution

Strategy. Separate eventual decay from the shape created by two modes with opposite coefficients.

Step 1: Solve for the two coefficients. The characteristic equation is $(r + 2)(r + 3) = 0$, so $y = Ae^{-2t} + Be^{-3t}$. The data give $A + B = 0$ and $-2A - 3B = 1$. Hence

$$y = e^{-2t} - e^{-3t}.$$

Its value and derivative at zero are 0 and 1, and both exponential terms satisfy the equation.

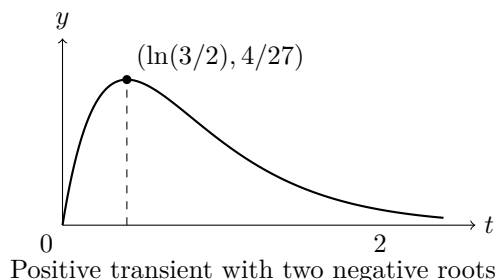
Step 2: Prove positivity and decay. For $t > 0$, $y = e^{-3t}(e^t - 1) > 0$. Both exponential terms tend to zero, so the positive response eventually decays to zero. Positivity does not require both mode coefficients to be positive.

Step 3: Locate and evaluate the only peak. The derivative factors as

$$y' = e^{-3t}(3 - 2e^t).$$

It is positive before $t_m = \ln(3/2)$ and negative afterward. At that time $e^{-t_m} = 2/3$, so

$$t_m = \ln(3/2), \quad y(t_m) = 4/9 - 8/27 = 4/27.$$



Step 4: Check the total area independently. Direct integration gives $\int_0^\infty y \, dt = 1/2 - 1/3 = \boxed{1/6}$. Alternatively, integrate the equation on $[0, T]$:

$$y'(T) - y'(0) + 5(y(T) - y(0)) + 6 \int_0^T y \, dt = 0.$$

The explicit formula verifies $y(T), y'(T) \rightarrow 0$ and convergence of the integral. Passing to the limit gives $-1 + 6 \int_0^\infty y \, dt = 0$, confirming the result.

Negative roots control the eventual limit of each mode. Their combination can initially increase: this solution starts at zero with positive slope, reaches a peak, and only then decreases.