

Question 2

Consider the family of initial-value problems

$$y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = v.$$

Call a solution forward bounded if $|y(t)|$ is bounded for $t \geq 0$.

Tasks

1. Find the roots and express the solution coefficients in terms of v .
2. Determine exactly which initial slope produces a forward-bounded solution. State its limit.
3. Replace that slope by $v = -2 + \varepsilon$, where $0 < \varepsilon < 1$. Find when the growing and decaying terms have equal magnitude, and determine the eventual behavior.
4. For $\varepsilon = 0.01$, calculate that time to three decimal places and sketch the perturbed and unperturbed solutions on $[0, 2]$. Explain why a small initial error can change the long-time conclusion.

Question 2 – Solution

Strategy. Express the initial data in the two modes and isolate the coefficient of the growing exponential.

Step 1: Resolve the data into modes. The characteristic polynomial is $(r - 3)(r + 2)$, so $y = Ae^{3t} + Be^{-2t}$. From $A + B = 1$ and $3A - 2B = v$,

$$A = (v + 2)/5, \quad B = (3 - v)/5.$$

The nonzero root separation makes this data system invertible.

Step 2: Remove the growing mode exactly. If $A \neq 0$, the term Ae^{3t} dominates and $|y| \rightarrow \infty$. Thus forward boundedness holds exactly when $A = 0$, or $v = -2$. The selected solution is $y = e^{-2t}$ and tends to zero.

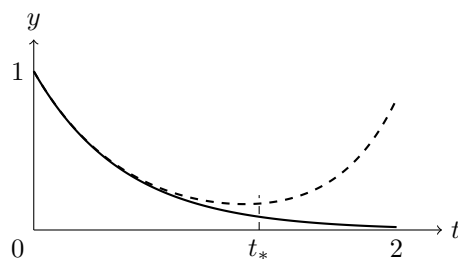
Step 3: Track a small slope perturbation. For $v = -2 + \varepsilon$,

$$y_\varepsilon = \frac{\varepsilon}{5}e^{3t} + \frac{5 - \varepsilon}{5}e^{-2t}.$$

Both terms are positive. They are equal precisely when $e^{5t} = (5 - \varepsilon)/\varepsilon$, giving

$$t_* = \frac{1}{5} \ln \frac{5 - \varepsilon}{\varepsilon}.$$

This is positive for the stated range of ε . Eventually $y_\varepsilon \rightarrow +\infty$, although its initial slope is negative.



Solid: $v = -2$; dashed: $v = -1.99$

Step 4: Quantify the delayed disagreement. For $\varepsilon = 0.01$, $t_* = (\ln 499)/5 \approx 1.243$. The exact difference is

$$y_\varepsilon - e^{-2t} = \frac{\varepsilon}{5}(e^{3t} - e^{-2t}).$$

It vanishes initially and is small with ε on any fixed finite interval, but grows without bound as $t \rightarrow \infty$ for every fixed positive ε . Forward boundedness here requires exact cancellation of a mode; it is not robust to a nonzero slope error. The time t_* compares mode magnitudes and is not the time of the trajectory's minimum.