

Question 1

Consider

$$6y'' + y' - 2y = 0, \quad y(1) = 7, \quad y'(1) = 0.$$

For a constant-coefficient homogeneous second-order equation with distinct real characteristic roots r_1, r_2 , the general solution is $C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

Tasks

1. Substitute $y = e^{rt}$ to derive the characteristic equation. Find and verify both roots.
2. Write the general solution using exponentials based at $t = 1$, then determine its constants from the initial data.
3. Verify the resulting function in the original equation and both initial conditions. Explain why it is the unique solution on $\mathbb{R}$.
4. Classify the stationary point at $t = 1$ and determine the limits as $t \rightarrow \infty$ and $t \rightarrow -\infty$. Explain why zero initial slope does not mean a constant solution.

Question 1 – Solution

Strategy. Keep the leading coefficient in the characteristic polynomial and choose a shifted exponential basis to simplify the data.

Step 1: Derive, factor and check the polynomial. Substitution gives $(6r^2 + r - 2)e^{rt} = 0$. Since $e^{rt} \neq 0$,

$$6r^2 + r - 2 = (2r - 1)(3r + 2) = 0, \quad \boxed{r_1 = 1/2, \quad r_2 = -2/3}.$$

Direct evaluation gives $6(1/2)^2 + 1/2 - 2 = 0$ and $6(-2/3)^2 - 2/3 - 2 = 0$.

Step 2: Use the actual initial time. A shifted basis only rescales the arbitrary constants, so write

$$y = Ae^{(t-1)/2} + Be^{-2(t-1)/3}.$$

The data give $A + B = 7$ and $A/2 - 2B/3 = 0$. Thus $A = 4$, $B = 3$, and

$$\boxed{y = 4e^{(t-1)/2} + 3e^{-2(t-1)/3}}.$$

Step 3: Verify the equation and uniqueness. Each exponential contributes its coefficient times $6r^2 + r - 2$ to the residual, so their sum solves the original equation. At 1, its value is $4 + 3 = 7$ and its derivative is $2 - 2 = 0$. After division by 6, all coefficients are continuous on $\mathbb{R}$. The linear initial-value theorem therefore gives uniqueness there.

Step 4: Interpret the initial balance of modes. Differentiating twice gives

$$y'' = e^{(t-1)/2} + \frac{4}{3}e^{-2(t-1)/3} > 0.$$

Thus y' is strictly increasing, crosses zero at 1, and $\boxed{y(1) = 7}$ is the unique global minimum. The positive-root mode gives $y \rightarrow +\infty$ as $t \rightarrow \infty$; the negative-root mode gives $y \rightarrow +\infty$ as $t \rightarrow -\infty$.

Zero slope at one instant is a balance between two changing modes. Indeed the equation at 1 gives $y''(1) = 7/3$, not zero. A constant solution would have to satisfy $-2y = 0$ and could not have value 7.