

Question 9

A 1 kg mass with $c = 2 \text{ N s/m}$ and $k = 5 \text{ N/m}$ obeys $x'' + 2x' + 5x = F(t)$ in SI units. An actuator must move it from $x(0) = 0.10 \text{ m}$, $x'(0) = 0$ to $x(1) = x'(1) = 0$, then turn off for $t > 1$. Before $t = 0$, it is held motionless at $x = 0.10 \text{ m}$ by a constant force.

Tasks

1. Find the unique cubic trajectory satisfying the four prescribed endpoint data on $[0, 1]$.
2. Compute its required force. Examine the force and acceleration when it joins the held state at zero and the rest state at one.
3. Instead require zero acceleration at both endpoints as well. Find the unique polynomial trajectory of degree at most five satisfying all six conditions.
4. Compute the new force and verify that it joins the pre-motion holding force and the zero post-motion force continuously. Show that the mass remains at rest after one second.

Question 9 – Solution

Strategy. Position and velocity matching ensure a continuous state, but continuous actuator force also requires compatible acceleration.

Step 1: Fit the cubic. Write $x = a_0 + a_1t + a_2t^2 + a_3t^3$. The initial conditions give $a_0 = 0.10$, $a_1 = 0$; the final conditions give $a_2 + a_3 = -0.10$, $2a_2 + 3a_3 = 0$. Hence

$$x_c = 0.10(1 - 3t^2 + 2t^3).$$

The nonsingular two-equation system makes this cubic unique.

Step 2: Check the actuator joins. Differentiating gives $x'_c = 0.60(t^2 - t)$ and $x''_c = 1.20t - 0.60$. Thus

$$F_c = x''_c + 2x'_c + 5x_c = t^3 - 0.30t^2 - 0.10 \quad \text{N.}$$

The holding force before zero is $5(0.10) = 0.50$ N. Yet $F_c(0) = -0.10$ N and $F_c(1) = 0.60$ N, so both joins require force jumps. Acceleration jumps from zero to -0.60 initially and from 0.60 to zero finally. The joined trajectory is C^1 and piecewise C^2 , not globally C^2 .

Step 3: Add acceleration matching. For a polynomial of degree at most five, the initial conditions fix $a_0 = 0.10$, $a_1 = a_2 = 0$. The endpoint conditions become

$$a_3 + a_4 + a_5 = -0.10, \quad 3a_3 + 4a_4 + 5a_5 = 0, \quad 6a_3 + 12a_4 + 20a_5 = 0.$$

Solving gives $a_3 = -1$, $a_4 = 1.5$, $a_5 = -0.6$, uniquely. Therefore

$$x_q = 0.10(1 - 10t^3 + 15t^4 - 6t^5).$$

Step 4: Verify force continuity and rest. Substitution into the model yields

$$F_q = 0.50 - 6t + 12t^2 - 5t^3 + 1.5t^4 - 3t^5 \quad \text{N.}$$

It has $F_q(0) = 0.50$ and $F_q(1) = 0$, matching the constant forces outside the motion interval. All six trajectory conditions hold, so the joined motion is C^2 . For $t > 1$, zero forcing and zero value and slope at one have the unique solution $x = 0$. The improvement is continuous force and acceleration; no claim of a force-minimizing trajectory is needed.