

Question 8

A 1 kg mass is connected to a moving support by a spring with $k = 9 \text{ N/m}$ and a parallel viscous damper with $c = 2 \text{ N s/m}$. Let $b(t) = B \cos \omega t$ be support displacement and $x(t)$ the mass's absolute displacement, measured in the same positive direction about equilibrium. Here $B > 0$ and $\omega > 0$. Spring and damping forces depend on relative displacement and relative velocity.

Tasks

1. Derive the equations for absolute displacement x and relative displacement $z = x - b$.
2. Find the steady amplitude ratios X/B and Z/B , where X and Z are the amplitudes of x and z .
3. Determine exactly when the absolute amplitude is smaller than the support amplitude, including the equality frequency.
4. Find the low- and high-frequency limits of both ratios. Explain why good isolation of the mass does not imply small relative spring deformation.

Question 8 – Solution

Strategy. The damper responds to velocity relative to the support. Distinguish motion of the mass from deformation of the suspension.

Step 1: Derive the two models. Newton's law gives $x'' = -9(x - b) - 2(x' - b')$. Therefore

$$x'' + 2x' + 9x = 2b' + 9b, \quad \boxed{z'' + 2z' + 9z = -b'' = B\omega^2 \cos \omega t.}$$

The inertial term in the relative equation comes from $x'' = z'' + b''$; it is not an extra spring force.

Step 2: Compute both amplitude ratios. Let $\Delta = (9 - \omega^2)^2 + 4\omega^2$. Matching sine and cosine in the relative equation gives

$$z = A \cos \omega t + D \sin \omega t, \quad A = \frac{B\omega^2(9 - \omega^2)}{\Delta}, \quad D = \frac{2B\omega^3}{\Delta}.$$

Thus $Z/B = \omega^2/\sqrt{\Delta}$. Since $x = z + B \cos \omega t$, its cosine coefficient is $B + A$, not its amplitude $B + Z$. Simplifying $(B + A)^2 + D^2$ gives

$$\boxed{\frac{X}{B} = \frac{\sqrt{81 + 4\omega^2}}{\sqrt{\Delta}}, \quad \frac{Z}{B} = \frac{\omega^2}{\sqrt{\Delta}}.}$$

The homogeneous roots $-1 \pm 2\sqrt{2}i$ ensure the initial-data transients decay.

Step 3: Find the isolation threshold. Since $\Delta > 0$, $X/B < 1$ exactly when

$$\Delta - (81 + 4\omega^2) = \omega^2(\omega^2 - 18) > 0.$$

For $\omega > 0$, this means $\boxed{\omega > 3\sqrt{2} \text{ s}^{-1}}$. Equality holds at $3\sqrt{2}$; below that positive frequency the absolute motion is amplified.

Step 4: Interpret the limiting motions. As $\omega \rightarrow 0^+$, $X/B \rightarrow 1$ and $Z/B \rightarrow 0$: the mass follows the slowly moving support. As $\omega \rightarrow \infty$, $X/B \rightarrow 0$ but $Z/B \rightarrow 1$. In fact $A/B \rightarrow -1$ and $D/B \rightarrow 0$, so $z \approx -b$ while $x \approx 0$. An almost stationary mass can still have relative deformation comparable to the base motion.

