

Question 7

A mass-spring-damper system has $m = 2$ kg, $c = 4$ N s/m and $k = 20$ N/m. Forcing is $F_0 \cos \omega t$, where $F_0 > 0$ is fixed and $\omega > 0$. Consider its steady periodic response. Let X be displacement amplitude and V be velocity amplitude.

Tasks

1. Derive the cosine and sine coefficients of the steady response and express X and V in terms of ω .
2. Find the frequency that maximizes displacement amplitude and its maximum value.
3. Independently find the frequency that maximizes velocity amplitude and its maximum value.
4. At the velocity-maximizing frequency, determine the displacement phase relative to the applied force. Explain why a request for the 'resonant frequency' needs a specified response quantity.

Question 7 – Solution

Strategy. Different measured quantities introduce different frequency factors; optimize their amplitudes separately.

Step 1: Match harmonic coefficients. Set $x = A \cos \omega t + B \sin \omega t$. The equation $2x'' + 4x' + 20x = F_0 \cos \omega t$ gives

$$(20 - 2\omega^2)A + 4\omega B = F_0, \quad -4\omega A + (20 - 2\omega^2)B = 0.$$

Writing $\Delta = (20 - 2\omega^2)^2 + 16\omega^2 > 0$, we obtain

$$A = \frac{F_0(20 - 2\omega^2)}{\Delta}, \quad B = \frac{4F_0\omega}{\Delta}, \quad \boxed{X = \frac{F_0}{\sqrt{\Delta}}, \quad V = \omega X.}$$

The homogeneous roots are $-1 \pm 3i$, so all initial-data transients decay and this periodic response is the long-time motion.

Step 2: Maximize displacement. Complete the square:

$$\Delta = 4[(\omega^2 - 8)^2 + 36].$$

It has its unique minimum for $\omega > 0$ at $\omega_X = 2\sqrt{2}\text{s}^{-1}$. Therefore $\boxed{X_{\max} = F_0/12}$ in meters when F_0 is expressed in newtons.

Step 3: Maximize velocity. Divide the amplitude denominator by ω :

$$V = \frac{F_0}{\sqrt{(20/\omega - 2\omega)^2 + 16}}.$$

This is largest exactly when $20/\omega - 2\omega = 0$, giving

$$\boxed{\omega_V = \sqrt{10}\text{s}^{-1}, \quad V_{\max} = F_0/4}$$

in meters per second when F_0 is in newtons. The spring and inertial contributions to this denominator cancel there.

Step 4: Interpret the phase and the two maxima. At $\omega = \sqrt{10}$, $A = 0$ and $B = F_0/(4\sqrt{10}) > 0$. Thus $x = [F_0/(4\sqrt{10})] \sin(\sqrt{10}t)$, a displacement lag of $\pi/2$ behind the cosine force. Its displacement amplitude is smaller than $F_0/12$, even though its velocity amplitude is largest. The displacement peak, velocity peak and undamped natural frequency need not all coincide; the response quantity must be named.