

Question 6

An undamped 1 kg mass on a 25 N/m spring is initially at rest at equilibrium. For $t \geq 0$, apply the force $2 \cos 4t$ N.

Tasks

1. Derive and solve the forced initial-value problem.
2. Express the response as a product that displays beats. Identify the carrier angular frequency and the period of the nonnegative amplitude envelope.
3. At $t = 2\pi$ s, find both displacement and velocity. Does this mean the mass stays at rest afterward? Justify your answer using the equation.
4. Replace the forcing by $2 \cos 5t$ N and find the new zero-data response. Explain the difference between beats and exact resonance.

Question 6 – Solution

Strategy. The homogeneous correction needed to fit rest data creates the nearby-frequency interference; exact resonance requires a different particular form.

Step 1: Solve the nonresonant problem. Newton's law gives $x'' + 25x = 2 \cos 4t$, $x(0) = x'(0) = 0$. Since $25 - 16 = 9$, a particular response is $(2/9) \cos 4t$. Fitting zero data gives

$$x = \frac{2}{9}(\cos 4t - \cos 5t).$$

Its second derivative plus $25x$ equals $2 \cos 4t$, and both initial data vanish.

Step 2: Separate the two time scales. The cosine-difference identity gives

$$x = \frac{4}{9} \sin(9t/2) \sin(t/2).$$

The carrier angular frequency is $9/2 \text{ s}^{-1}$. The nonnegative envelope is $A(t) = (4/9)|\sin(t/2)| \text{ m}$, with period $2\pi \text{ s}$. Its zeros occur at $t = 2n\pi$, and its largest value is $4/9 \text{ m}$, attained by x at $t = \pi$. The envelope period uses the absolute value; the signed slow sine has period 4π .

Step 3: Distinguish a rest state from remaining at rest. Here $x' = \frac{2}{9}(-4 \sin 4t + 5 \sin 5t)$, so $x(2\pi) = x'(2\pi) = 0$. Yet the applied force then equals 2 N ; the equation gives $x''(2\pi) = 2 \text{ m/s}^2$. The mass immediately accelerates again. Reaching the equilibrium rest state while a nonzero force persists does not keep it there.

Step 4: Solve at exact resonance. For $x'' + 25x = 2 \cos 5t$, use $x_p = At \sin 5t$. Its residual is $10A \cos 5t$, so $A = 1/5$ and the data are already zero:

$$x_{\text{res}} = \frac{1}{5}t \sin 5t.$$

At $t_n = (\pi/2 + 2n\pi)/5$, it equals $t_n/5 \rightarrow \infty$, whereas the beating response is bounded by $4/9$. Beats arise from interference of distinct frequencies; resonance produces a growing factor when the forcing matches the natural frequency.

