

Question 5

A 1 kg mass on a spring with $k = 4 \text{ N/m}$ is released from $x(0) = a = 0.01 \text{ m}$, with $x'(0) = 0$. Compare viscous damping coefficients $c = 4$ and $c = 5 \text{ Ns/m}$; write their responses as x_4, x_5 .

Tasks

1. Determine the damping regimes and find both exact responses.
2. Prove that each response stays positive and strictly decreases for $t > 0$.
3. Prove $x_5(t) > x_4(t)$ for every $t > 0$. You may form the equation for $w = x_5 - x_4$ and derive its zero-data integral response.
4. Explain the implication for the first time the displacement reaches any fixed level ε with $0 < \varepsilon < a$. For general $c > 4$, examine the slow characteristic root as $c \rightarrow \infty$.

Question 5 – Solution

Strategy. Compare complete release responses, rather than assuming a larger damping force always gives a faster return.

Step 1: Solve both release problems. For $c = 4$, the repeated root is -2 , giving critical damping. For $c = 5$, roots $-1, -4$ give overdamping. Fitting the data yields

$$x_4 = a(1 + 2t)e^{-2t}, \quad x_5 = \frac{a}{3}(4e^{-t} - e^{-4t}).$$

Their characteristic forms verify the equations; both have value a and slope zero at the release.

Step 2: Check positivity and decrease. The first expression is positive, and the second is positive because $4e^{-t} > e^{-4t}$. Their derivatives are

$$x_4' = -4ate^{-2t} < 0, \quad x_5' = \frac{4a}{3}(e^{-4t} - e^{-t}) < 0 \quad (t > 0).$$

Both tend to zero.

Step 3: Prove the comparison for every time. Applying $D^2 + 5D + 4$ to $w = x_5 - x_4$ gives

$$w'' + 5w' + 4w = -x_4', \quad w(0) = w'(0) = 0.$$

Variation of parameters with e^{-t}, e^{-4t} yields

$$w(t) = \int_0^t \frac{e^{-(t-s)} - e^{-4(t-s)}}{3} [-x_4'(s)] ds.$$

For $0 < s < t$, both factors are strictly positive. Hence $\boxed{x_5(t) > x_4(t)}$ for every $t > 0$. This integral proof avoids relying on a finite plot or only on large-time rates.

Step 4: Interpret settling and heavy damping. Each response reaches any $\varepsilon \in (0, a)$ exactly once, by continuity and strict decrease. At the time $x_4 = \varepsilon$, one still has $x_5 > \varepsilon$, so the more heavily damped response reaches the level later. For general $c > 4$ in this SI model,

$$r_{\text{slow}} = \frac{-c + \sqrt{c^2 - 4mk}}{2m} = \frac{-2k}{c + \sqrt{c^2 - 4mk}} \rightarrow 0^-.$$

The release excites this mode with coefficient $-ar_{\text{fast}}/(r_{\text{slow}} - r_{\text{fast}}) > 0$. Excessive damping therefore introduces a slow return, with $r_{\text{slow}} \sim -k/c$; damping strength alone is not a measure of settling speed.