

Question 4

A critically damped system has $m = 1$ kg, $c = 6$ N s/m and $k = 9$ N/m. Its initial data are $x(0) = a > 0$ and $x'(0) = v$, with a in meters and v in meters per second. There is no forcing.

Tasks

1. Derive and verify the response for arbitrary a, v .
2. Classify exactly which initial velocities cause a finite equilibrium crossing. Find the crossing time when one exists.
3. Classify exactly which initial velocities make the displacement stay positive and decrease monotonically toward zero.
4. For $a = 0.10$ m and $v = -0.60$ m/s, find the crossing and the subsequent minimum. Use the result to assess the claim that critical damping always prevents overshoot.

Question 4 – Solution

Strategy. Critical damping fixes the repeated root, but the sign of the linear factor still depends on the initial velocity.

Step 1: Fit the repeated-root solution. The characteristic polynomial is $(r + 3)^2$. Thus

$$x(t) = e^{-3t}[a + (v + 3a)t], \quad x'(t) = e^{-3t}[v - 3(v + 3a)t].$$

Here the numerical rate 3 has units s^{-1} . The two data follow immediately, and substitution gives $x'' + 6x' + 9x = 0$.

Step 2: Classify crossings. Since $e^{-3t} > 0$, a positive-time zero occurs exactly when $v + 3a < 0$. In that case it is unique and

$$t_c = \frac{a}{-v - 3a}.$$

The linear factor changes sign there, so this is a crossing, not a touch. If $v \geq -3a$, the factor remains positive for all finite $t \geq 0$, including the equality case $x = ae^{-3t}$.

Step 3: Classify monotone positive return. For $v \geq -3a$, the displacement is positive. Its derivative is nonpositive for all $t \geq 0$ exactly when $v \leq 0$, because $v - 3(v + 3a)t$ then begins nonpositive and decreases. Thus the required range is

$$-3a \leq v \leq 0.$$

For $v > 0$, the mass first moves farther from equilibrium. For $v < -3a$, it crosses equilibrium and cannot be a positive monotone return.

Step 4: Exhibit overshoot. For the stated data, $x = 0.10e^{-3t}(1 - 3t)$ and $x' = e^{-3t}(-0.60 + 0.90t)$. Therefore

$$t_c = 1/3 \text{ s}, \quad t_{\min} = 2/3 \text{ s}, \quad x_{\min} = -0.10e^{-2} \text{ m}.$$

The derivative changes from negative to positive at the minimum. Critical damping eliminates sustained oscillations, but sufficiently negative initial velocity still causes one overshoot. The comparison uses $\tau = 3t$ and normalized displacement x/a .

