

Question 3

A mass-spring-damper system has $m = 1$ kg, $c = 2$ Ns/m and $k = 5$ N/m. It starts at equilibrium with velocity 1 m/s in the positive direction, with no external forcing. Define

$$E(t) = \frac{1}{2}mx'(t)^2 + \frac{1}{2}kx(t)^2.$$

Tasks

1. Find the displacement and velocity, including the damping regime.
2. Derive the energy balance directly from the equation. Does decreasing energy imply that $|x|$ must decrease from the start?
3. Find the first positive turning time, its displacement, and the energy dissipated up to that time.
4. Explain why $E' = 0$ at a turning point does not mean damping has ceased permanently. Find the total energy dissipated as $t \rightarrow \infty$.

Question 3 – Solution

Strategy. Track energy in both position and velocity. Displacement can grow while total mechanical energy falls.

Step 1: Solve the free motion. The equation is $x'' + 2x' + 5x = 0$, with roots $-1 \pm 2i$, so the motion is underdamped. The data give

$$x = \frac{1}{2}e^{-t} \sin 2t, \quad x' = e^{-t}[\cos 2t - \frac{1}{2} \sin 2t].$$

These expressions have $x(0) = 0$, $x'(0) = 1$ and residual zero.

Step 2: Derive the dissipation law. Multiplying the equation by x' gives

$$E' = x'(mx'' + kx) = -cx'^2 = \boxed{-2x'^2 \leq 0}.$$

Nevertheless, $x'(0) = 1$, so x and $|x|$ initially increase from zero. Kinetic energy is partly converted into spring energy while the damper removes energy. Monotonic total energy is not monotonic displacement.

Step 3: Find the first turning point. The first root of $x' = 0$ satisfies $\tan 2t = 2$, in $0 < 2t < \pi/2$. Set $a = \frac{1}{2} \arctan 2$. Then

$$\boxed{t_* = a, \quad x(t_*) = e^{-a}/\sqrt{5} \text{ m.}}$$

The velocity changes from positive to negative, so this is a positive maximum. Initially $E(0) = 1/2 \text{ J}$. At the turning point,

$$E(a) = \frac{5}{2}x(a)^2 = \frac{1}{2}e^{-2a} \text{ J}, \quad \boxed{E(0) - E(a) = \frac{1}{2}(1 - e^{-2a}) \text{ J.}}$$

Step 4: Interpret zero instantaneous loss. At a turning point, $x' = 0$, so damper power is instantaneously zero. But $x \neq 0$ there and $x'' = -5x \neq 0$, so motion resumes immediately. The displayed response and velocity both tend to zero; hence $E \rightarrow 0$ and total dissipated energy is $1/2 \text{ J}$. In the figure, $X = \sqrt{k/m}x$ and $V = x'$ both have velocity units, and $E = m(X^2 + V^2)/2$.

