

Question 10

For $t \geq 0$, solve

$$y'' + y = e^{-t^2}, \quad y(0) = y'(0) = 0.$$

Define the convergent integrals $C = \int_0^\infty e^{-s^2} \cos s \, ds$ and $S = \int_0^\infty e^{-s^2} \sin s \, ds$. No special-function evaluations of C or S are required.

Tasks

1. Use variation of parameters to express the exact solution through two finite definite integrals, and verify its initial data and residual.
2. Show that the solution approaches the periodic function $P(t) = C \sin t - S \cos t$ in the sense that $y(t) - P(t) \rightarrow 0$.
3. Prove the quantitative estimate $|y(t) - P(t)| \leq e^{-t^2}/(2t)$ for every $t > 0$.
4. Prove $S > 0$ by pairing successive positive and negative half-waves of sine. Use this to show that y does not tend to zero, even though the forcing does.

Question 10 – Solution

Strategy. Keep the variation-of-parameters integrals exact, then bound their tails to identify the surviving oscillation.

Step 1: Write and check the exact response. With $W = 1$, the parameter derivatives are $u'_1 = -e^{-t^2} \sin t$ and $u'_2 = e^{-t^2} \cos t$. Hence

$$y(t) = \sin t \int_0^t e^{-s^2} \cos s \, ds - \cos t \int_0^t e^{-s^2} \sin s \, ds.$$

Equivalently, $y = \int_0^t \sin(t-s)e^{-s^2} \, ds$. Differentiating gives $y' = \int_0^t \cos(t-s)e^{-s^2} \, ds$ and $y'' = e^{-t^2} - y$; both initial data vanish.

Step 2: Isolate the tails. Absolute convergence follows from the integrability of e^{-s^2} (for $s \geq 1$, it is at most e^{-s}). Subtracting the full integrals gives

$$y(t) - P(t) = - \int_t^\infty \sin(t-s)e^{-s^2} \, ds.$$

Thus $|y - P| \leq \int_t^\infty e^{-s^2} \, ds \rightarrow 0$.

Step 3: Obtain an explicit error bound. For $s \geq t > 0$, $1 \leq s/t$. Consequently

$$|y - P| \leq \frac{1}{t} \int_t^\infty s e^{-s^2} \, ds = \boxed{\frac{e^{-t^2}}{2t}}.$$

Combining the tails before estimating avoids an unnecessary sum of two bounds.

Step 4: Prove a nonzero surviving oscillation. Absolute convergence permits pairing intervals. Their k th pair contributes

$$\int_0^\pi \sin u [e^{-(2k\pi+u)^2} - e^{-((2k+1)\pi+u)^2}] \, du > 0, \quad k \geq 0.$$

Therefore $S > 0$. Along $t = 2n\pi$, $y(t) \rightarrow -S$; along $t = (2n+1)\pi$, $y(t) \rightarrow S$. The solution has no limit and does not decay to zero. A decaying forcing can leave a persistent homogeneous oscillation.

