

Question 9

Let f be continuous on $[0, \pi]$. Seek C^2 solutions of

$$y'' + y = f(t), \quad y(0) = y(\pi) = 0.$$

Use variation of parameters; no general boundary-value theory is required.

Tasks

1. Write every solution satisfying the first endpoint condition, using its free initial slope A .
2. Derive a necessary and sufficient condition on f for the second endpoint condition. Explain uniqueness or nonuniqueness when the condition holds.
3. Apply the criterion to $f = 1$ and $f = \cos t$, and find all solutions in the latter case.
4. For $f = \cos t$, impose the additional condition $\int_0^\pi y(t) dt = 0$. Determine whether it selects a unique solution and find it.

Question 9 – Solution

Strategy. Variation of parameters supplies all initial-value responses; evaluating the integral at the second endpoint reveals the obstruction.

Step 1: Keep the free slope. The homogeneous pair is $\cos t, \sin t$, with $W = 1$. Integrating the parameter derivatives from zero and imposing $y(0) = 0$ gives every candidate:

$$y(t) = A \sin t + \int_0^t \sin(t-s)f(s) ds, \quad y'(0) = A.$$

Differentiating this formula verifies the equation and shows no candidate has been lost.

Step 2: Evaluate the second endpoint. Since $\sin \pi = 0$ and $\sin(\pi - s) = \sin s$, the remaining condition is

$$\int_0^\pi \sin s f(s) ds = 0.$$

If it fails, no choice of A helps. If it holds, every real A works: there are infinitely many solutions, differing by $A \sin t$. Thus this endpoint pair never selects exactly one solution for a compatible forcing.

Step 3: Compare the two forcings. For $f = 1$, the integral is 2, so there is no solution. For $f = \cos t$, it is $\int_0^\pi \sin s \cos s ds = 0$. Evaluation of the response integral, or a direct residual check, gives

$$y(t) = (A + t/2) \sin t, \quad A \in \mathbb{R}.$$

Indeed, $(t \sin t/2)'' + t \sin t/2 = \cos t$, and both endpoints vanish.

Step 4: Apply the extra measurement. The needed integrals are $\int_0^\pi \sin t dt = 2$ and, by integration by parts, $\int_0^\pi t \sin t dt = \pi$. Thus

$$0 = 2A + \pi/2, \quad A = -\pi/4, \quad y(t) = (t/2 - \pi/4) \sin t.$$

The coefficient of A in this measurement is nonzero, so it selects exactly one member of the compatible family. This does not contradict the earlier nonuniqueness: the added integral condition supplies new information.