

Question 8

For $t \geq 0$, solve $y'' + y = g(t)$ with zero initial value and slope. First use the continuous finite-duration forcing

$$g_2(t) = \begin{cases} \sin 2t, & 0 \leq t \leq \pi, \\ 0, & t > \pi. \end{cases}$$

Later compare it with g_3 , defined in the same way using $\sin 3t$.

Tasks

1. Derive the integral response and solve explicitly for g_2 on $[0, \pi]$.
2. Find the response after π , checking the matching value and slope. Does turning off the forcing return the solution to rest?
3. For any continuous forcing supported in $[0, \pi]$, derive the two integral conditions equivalent to $y(t) = 0$ for every $t \geq \pi$.
4. Test g_3 against these conditions and find its response. Explain how a nonzero forcing can leave no later motion.

Question 8 – Solution

Strategy. After the forcing ends, its two weighted integrals become the coefficients of the remaining homogeneous oscillation.

Step 1: Solve while the first forcing acts. Variation of parameters gives $y(t) = \int_0^t \sin(t-s)g(s) ds$. For g_2 on $[0, \pi]$, evaluation gives

$$y_2(t) = \frac{2 \sin t - \sin 2t}{3}.$$

Its value and slope at zero vanish, and $y_2'' + y_2 = \sin 2t$, independently verifying the integral evaluation.

Step 2: Match at the shutoff time. At π , $y_2 = 0$ and $y_2' = -4/3$. The subsequent homogeneous response is

$$y_2(t) = -\frac{4}{3} \sin(t - \pi) = \frac{4}{3} \sin t, \quad t \geq \pi.$$

It matches both data. Since $g_2(\pi) = 0$, $y'' = g_2 - y$ also matches. The resulting function is a classical C^2 solution, but it does not return to rest.

Step 3: Derive the cancellation conditions. For any such forcing and $t \geq \pi$,

$$y(t) = \sin t \int_0^\pi \cos s g(s) ds - \cos t \int_0^\pi \sin s g(s) ds.$$

Independence of sine and cosine shows that the response vanishes identically after π exactly when both displayed integrals are zero. Cancellation of just one moment is insufficient.

Step 4: Construct a return to rest. For g_3 , product-to-sum identities give $\int_0^\pi \cos s \sin 3s ds = 0$ and $\int_0^\pi \sin s \sin 3s ds = 0$. Explicitly,

$$y_3(t) = \begin{cases} (3 \sin t - \sin 3t)/8, & 0 \leq t \leq \pi, \\ 0, & t \geq \pi. \end{cases}$$

On the first interval it has residual $\sin 3t$ and zero initial data; at π its value and slope are zero. The forcing's signed contributions cancel both final-state components, despite a nonzero response during the forcing interval.

