

Question 7

On $t \geq 0$, consider the family

$$y'' - y = \frac{1}{1+t^2}, \quad y(0) = 0, \quad y'(0) = v.$$

Write $f(t) = 1/(1+t^2)$ and $I = \int_0^\infty e^{-s} f(s) ds$. An answer containing a convergent definite integral is exact.

Tasks

1. Use variation of parameters to express y_v as an integral. Prove that I converges.
2. Find the only possible value of v for which y_v is bounded on $[0, \infty)$.
3. For that value, rewrite the response using the positive kernel $K(t, s) = \sinh(\min\{t, s\})e^{-\max\{t, s\}}$. Prove boundedness and determine its sign for $t > 0$.
4. Prove that the bounded response tends to zero, and determine the long-time behavior when the initial slope differs from the selected value.

Question 7 – Solution

Strategy. Cancel the growing mode using a convergent integral, then prove the resulting tail expression is bounded.

Step 1: Construct the family. Variation of parameters with e^t, e^{-t} gives

$$y_v = v \sinh t + \int_0^t \sinh(t-s)f(s) ds.$$

Since $0 < f \leq 1$, $0 < I \leq \int_0^\infty e^{-s} ds = 1$, with strict inequality $I < 1$.

Step 2: Identify the necessary slope. Expanding the hyperbolic sine yields

$$e^{-t}y_v = \frac{v}{2}(1 - e^{-2t}) + \frac{1}{2} \int_0^t e^{-s} f(s) ds - \frac{1}{2} e^{-2t} \int_0^t e^s f(s) ds.$$

The last term has magnitude at most $(e^{-t} - e^{-2t})/2$. Thus $e^{-t}y_v \rightarrow (v + I)/2$. Boundedness requires $v_* = -I$.

Step 3: Prove sufficiency and sign. Splitting the integral at $s = t$ and substituting $v = -I$ gives

$$y_*(t) = - \int_0^\infty K(t,s)f(s) ds, \quad K(t,s) = \begin{cases} e^{-t} \sinh s, & s \leq t, \\ e^{-s} \sinh t, & s \geq t. \end{cases}$$

This identity follows by grouping the e^t and e^{-t} terms above. Moreover,

$$\int_0^\infty K(t,s) ds = e^{-t}(\cosh t - 1) + e^{-t} \sinh t = 1 - e^{-t}.$$

Therefore $|y_*| \leq 1 - e^{-t} < 1$. For $t > 0$, the kernel and forcing are positive for $s > 0$, so $y_*(t) < 0$.

Step 4: Prove decay and sensitivity. Given $\varepsilon > 0$, choose A with $f(s) \leq \varepsilon$ for $s \geq A$. For $t > A$, the integral over $[0, A]$ has magnitude at most $e^{-t} \int_0^A \sinh s ds \rightarrow 0$; the remaining magnitude is at most $\varepsilon \int_A^\infty K \leq \varepsilon$. Hence $y_* \rightarrow 0$. If $v = v_* + \delta$, then $y_v = y_* + \delta \sinh t$, so $y_v/e^t \rightarrow \delta/2$. It tends to $+\infty$ for $\delta > 0$ and $-\infty$ for $\delta < 0$.